Simple models of plasma turbulence

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Outline

- The Hasegawa-Mima model
  - ⇛ derivation
  - ⇛ comparison with 2D fluid
  - ⇛ classical drift wave
  - ⇛ energy spectrum
  - ⇛ coherent structures
- The Hasegawa-Wakatani model
  - ⇛ derivation
  - ⇛ classical drift wave
- Connection with geophysics
  - ⇛ Charney model for Rossby waves

The Hasegawa-Mima model

Properties of the Hasegawa-Mima model

- 2D equation for potential, without sources or sinks (freely decaying 2D turbulence)
- Generalised energy and Enstrophy conserved
- Contains a specific scale length, i.e. ρ_s
- For $k\rho_s >> 1$ similar to 2D incompressible Euler fluid.

Assumptions of the Hasegawa-Mima model

- Homogeneous background magnetic field

$$\mathbf{B} = B_0 \hat{z}$$

- Inhomogeneous plasma

$$n_0 = n_0(x)$$

- Cold ions (ion thermal balance equation dropped) and

$$T_i << T_e$$

- Adiabatic electrons

$$n = n_e = n_0(x) exp\left(\frac{e\phi}{T_e}\right)$$

- Also, a small parameter is introduced (strong magnetic field approximation)

$$\epsilon = \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \qquad \text{where} \quad \omega_{ci} = \frac{eB_0}{m}$$

Derivation of the equation

- Start with:
  - Ion continuity equation
$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$
  - Ion momentum balance equation
$$m_i n \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = -\nabla p_i + en(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{F} - \nabla \Pi$$
    Neglect the anisotropy in pressure tensor and friction
  - Electrostatic perturbations $\mathbf{E} = -\nabla\phi$
  - Cold ions $\nabla p \to 0$
- The ion momentum balance then yields
$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{e}{m}\nabla\phi + \frac{e}{m}\mathbf{u} \times \mathbf{B}$$

Derivation of the equation

➩ Assuming ε small, for the first approximation one can drop the d/dt from the momentum balance

$$-\frac{e}{m}\nabla\phi + \frac{e}{m}\mathbf{u}\times\mathbf{B} = 0 \Rightarrow \mathbf{u} = \mathbf{u}_E = -\nabla\phi\times\frac{\mathbf{B}}{B_0^2}$$

ExB drift

➩ Substituting $\mathbf{u}_E$ in the momentum balance, one has a next order correction

$$\frac{\partial}{\partial t}\mathbf{u}_E + (\mathbf{u}_E\cdot\nabla)\mathbf{u}_E = \frac{e}{m}\mathbf{u}\times\mathbf{B} \Rightarrow \mathbf{u} = \mathbf{u}_P = \frac{1}{\omega_{ci}B_0}\left[-\frac{\partial}{\partial t}\nabla_\perp\phi - (\mathbf{u}_E\cdot\nabla_\perp)\nabla_\perp\phi\right]$$

Polarisation drift

➩ Thus the total velocity is

$$\mathbf{u} = -\nabla\phi\times\frac{\mathbf{B}}{B_0^2} + \frac{1}{\omega_{ci}B_0}\left[-\frac{\partial}{\partial t}\nabla_\perp\phi - (\mathbf{u}_E\cdot\nabla_\perp)\nabla_\perp\phi\right]$$

Polarisation

$$\frac{|\mathbf{u}_p|}{|\mathbf{u}_E|} = \left|\frac{1}{\omega_{ci} B_0}\frac{\partial}{\partial t}(\nabla\phi)\right| \Big/ \left|\frac{\nabla\phi}{B}\right| = \frac{1}{\omega_c \tau} = \epsilon$$

➪The polarisation velocity is negligible against the ExB but it is nevertheless kept since in a uniform magnetic field

$$\nabla\cdot\mathbf{u}_E = \nabla\cdot\left[\frac{\nabla\phi\times\mathbf{B}}{B_0^2}\right] = 0 \quad \text{but} \quad \nabla\cdot\mathbf{u}_p \neq 0$$

➪It is the polarisation drift that allows for compression of the ions, i.e. the polarisation drift allows for the change in the ion density such that in the presence of a potential quasi-neutrality is satisfied.

Derivation of the equations

Using the continuity equation and adiabaticity

$$\frac{dlnn}{dt} + \nabla \cdot \mathbf{u} = 0 \qquad \frac{\tilde{n}}{n_0} = \frac{e\phi}{T_e}$$

and assuming a small potential eϕ/T <<1 such that

$$lnn = lnn_0 + ln\left[1 + \frac{\tilde{n}}{n_0}\right] = lnn_0 + ln\left[1 + \frac{e\phi}{T_e}\right] \approx lnn_0 + \frac{e\phi}{T_e}$$

we obtain

$$\frac{d}{dt}\left[\frac{e\phi}{T_e}\right] + \mathbf{u}_E \cdot \nabla lnn_0 + \nabla \cdot \mathbf{u}_P = 0$$

Note that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \approx \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla$$

polarisation is kept
under the divergence
but neglected
in the time derivative
d/dt

Derivation of the equation

- Substituting $\mathbf{u}_P$, we get

$$\frac{d}{dt}\left(\frac{1}{\omega_{ci}B_0}\nabla_\perp^2\phi - \frac{e\phi}{T_e}\right) + (\mathbf{u}_E\cdot\nabla)ln\frac{n_0}{\omega_{ci}} = 0$$

- Use normalisation

$$\omega_{ci}t \to t \quad \frac{x,y}{\rho_s} \to x,y \quad \frac{e\phi}{T_e} \to \phi \;\; where \;\; \rho_s = \sqrt{\frac{T_e}{m_i}}\cdot\frac{1}{\omega_{ci}}$$

To get

$$\frac{\partial}{\partial t}(\nabla^2\phi - \phi) - [(\nabla\phi\times\hat{z})\cdot\nabla]\left[\nabla^2\phi - ln\left(\frac{n_0}{\omega_{ci}}\right)\right] = 0$$

Hasegawa-Mima equation

Evolution in k-space: Wave-wave coupling

- Hasegawa-Mima equation

$$\frac{\partial}{\partial t}[\nabla^2\phi - \phi] - [\nabla\phi \times \hat{z}] \cdot \nabla \left[\nabla^2\phi - ln\left(\frac{n_0}{\omega_{ci}}\right)\right] = 0$$

quadratic in ϕ

- Assume

$$\phi = \frac{1}{2}\sum_{\mathbf{k}} [\phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + c.c]$$

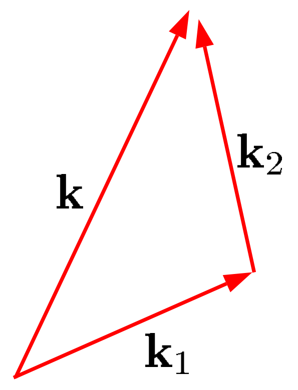
Then

$$\frac{d\phi_{\mathbf{k}}}{dt} + i\omega_{\mathbf{k}}\phi_{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2=0} \Lambda^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \phi^*_{\mathbf{k}_1} \phi^*_{\mathbf{k}_2}$$

with a matrix describing the coupling

$$\Lambda^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} = \frac{1}{2}\frac{1}{1+k^2}(\mathbf{k}_1 \times \mathbf{k}_2) \cdot \hat{z}[k_2^2 - k_1^2]$$

wave-wave coupling



Comparison with 2D fluid

The Hasegawa-Mima equation is similar to that of a 2D incompressible fluid

$$mn\frac{d\mathbf{u}}{dt} = -\nabla p \quad \text{with} \quad \nabla \cdot \mathbf{u} = 0$$

Because the 2D flow is incompressible, one can introduce a stream function ψ

$$\mathbf{u} = \hat{z} \times \nabla\psi \quad \text{where} \quad \psi = \psi(x, y) \Rightarrow \nabla \cdot \mathbf{u} = 0$$

Then taking the rotation of the equation of motion, one can eliminate the pressure gradient

$$mn\frac{d}{dt}(\nabla \times \mathbf{u}) = -\nabla \times \nabla p = 0$$

Substituting the velocity, one gets

$$\frac{\partial}{\partial t}\nabla^2\psi - (\nabla\psi \times \hat{z}) \cdot \nabla(\nabla^2\psi) = 0$$

Euler's equation for 2D fluid

Comparison with 2D fluid

- 2D incompressible fluid

$$\frac{\partial}{\partial t}\nabla^2\psi - (\nabla\psi \times \hat{z}) \cdot \nabla(\nabla^2\psi) = 0$$

vorticity

$$\left(\frac{\partial}{\partial t} - \nabla\psi \times \hat{z} \cdot \nabla\right) \nabla^2\psi = 0$$

- Hasegawa-Mima

$$\frac{\partial}{\partial t}[\nabla^2\phi - \phi] - (\nabla\phi \times \hat{z}) \cdot \nabla \left[\nabla^2\phi - \ln\left(\frac{n_0}{\omega_{ci}}\right)\right] = 0$$

$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right) \left[\nabla^2\phi - \phi - ln\left(\frac{n_0}{\omega_{ci}}\right)\right] = 0$$

potential vorticity

- Similar in structure, 2 differences

⇛ Hasegawa-Mima is not isotropic. The background gradient appears $\nabla ln(n_0/\omega_{ci})$

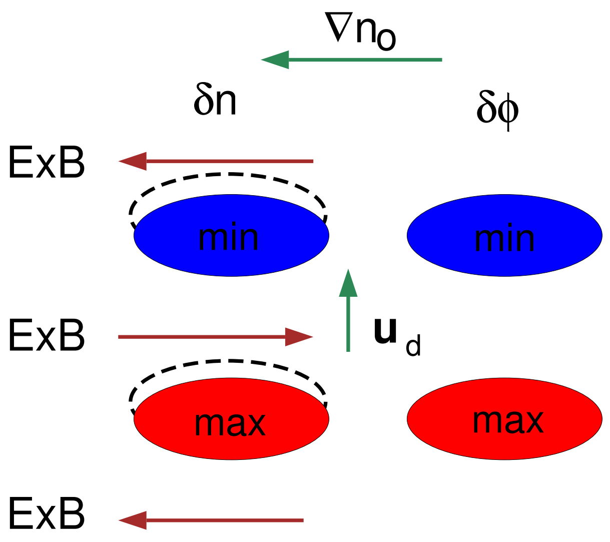
⇛ Hasegawa-Mima is not scale-invariant. Under the time derivative ϕ as well as $\nabla^2\phi$ appears. HM behaves like a 2D incompressible fluid only if

$$\phi/\nabla^2\phi \to 0 \quad or \quad k\rho_s >> 1$$

Wavelengths much smaller than ρ_s

Classical Drift Wave

- Assume a density perturbation
- Adiabatic electrons yield a potential perturbation $\phi \propto \delta n$
- ExB velocity around the max and min of $\delta\phi$
- ExB velocity shifts the δn max and min.
- If the electrons are adiabatic, there is no instability



- If the electrons are non-adiabatic (difference in phase between δn and $\delta\phi$) particles from the incide could move in region with max δn and particles from the outside could move in region with min $\delta n \Rrightarrow$ an instability can occur.

Drift waves in the Hasegawa-Mima equation

- We look for plane-wave solutions in the Hasegawa-Mima equation

$$\phi = \phi_k e^{i\mathbf{kx} - i\omega t}$$

- Then we can calculate the dispersion relation of the drift wave

$$\omega = \omega_k = -\frac{(\mathbf{k} \times \hat{z}) \cdot \nabla ln n_0}{1 + k^2} = \frac{\omega_*}{1 + k^2}$$

- At long wavelengths $k^2 << 1$ the wave propagates with speed

$$\mathbf{u}_d = \hat{z} \times \nabla ln n_0 \left(\frac{T_e}{eB} \right)$$ diamagnetic drift

and frequency

$$\omega_k = \mathbf{k} \cdot \mathbf{u}_d = \omega_*$$

- Since the Hasegawa-Mima equation is calculated with adiabatic electrons, there exists no instability. For an instability to occur, one has to include a forcing term (by hand) in the equation.

Conservation properties

In the Hasegawa-Mima model, there are two conserved quantities

$$W = \int dV \frac{\phi^2 + (\nabla\phi)^2}{2}$$ generalised energy

$$U = \int dV \frac{(\nabla\phi)^2 + (\nabla^2\phi)^2}{2}$$ generalised enstrophy

This indicates that there exists two types of inertia ranges in which the quantities W and U have different cascade properties.

For a 2D incompressible fluid

$$W = \int dV \frac{1}{2}(\nabla\phi)^2 \qquad U = \int dV \frac{1}{2}(\nabla^2\phi)^2$$

⇨ For $k\rho_s$ >>1 the conserved quantities in Hasegawa-Mima and in a 2D incompressible fluid are the same.

Kolmogorov -Kraichnan spectra

- Discussed in the first two talks (briefly repeated here)

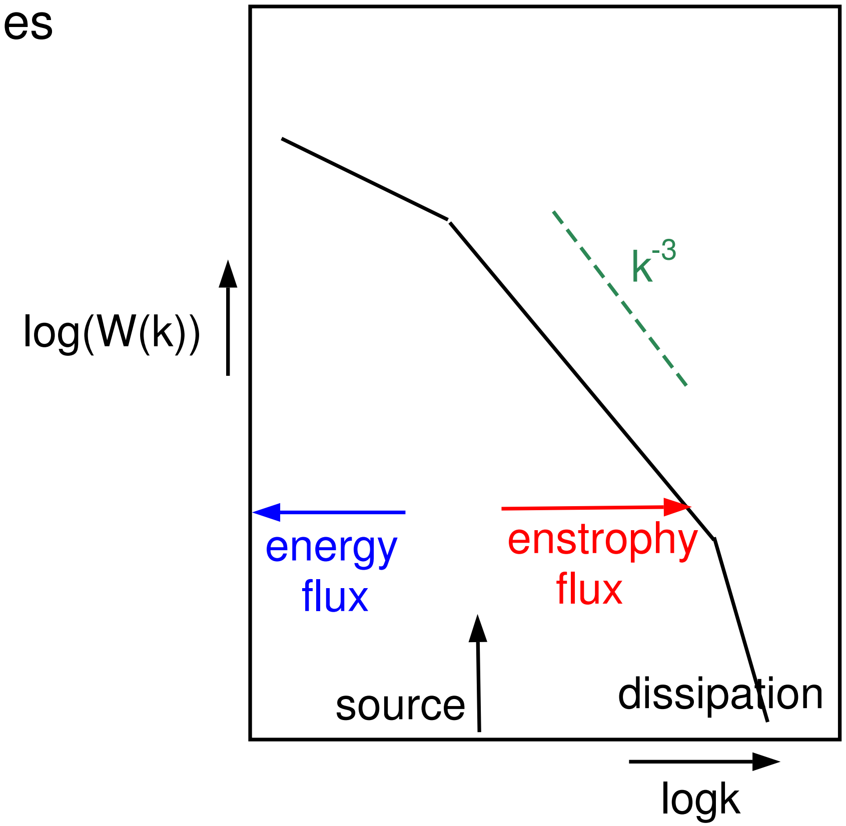
⇛ 3D turbulence, energy cascades to small scales where it is dissipated

$$W(k) = c\epsilon^{-2/3}k^{-5/3}$$

⇛ 2D turbulence, enstrophy cascades to small scales

$$W(k) = c'\eta^{2/3}k^{-3}$$

- Hasegawa-Mima close to the latter case.



Assumptions of Kolmogorov-Kraichnan

- Isotropy $W(\mathbf{k}) \Rightarrow W(k)$

Note $\int W(k)dk = total \;\; energy \quad kW(k) \propto v_k^2$

- Inertial range (no source or sink) ⇛ A stationary spectrum requires a wave vector independent enstrophy flux η

$$\eta = k^2 v_k^2 \frac{1}{\tau}$$

enstrophy (→ $k^2 v_k^2$), timescale of decay (→ τ)

- Local (in k space) assumption

$$\frac{1}{\tau} = k v_k$$

⇛ Results

$$\eta = k^3 v_k^3 = k^3 [kW(k)]^{3/2} \Rightarrow W(k) = c\eta^{2/3} k^{-3}$$

Problems in direct application of Kolmogorov-Kraichnan

- There is an additional length scale in this problem, i.e. ρ_s
  The spectrum is different for

$$k\rho_s << 1 \quad and \quad k\rho_s >> 1$$

- $W_k = (1+k^2)|\phi_k|^2 \neq v_k^2 = k^2|\phi_k|^2$ and $U_k = (1+k^2)k^2|\phi_k|^2$

- Isotropy is not guaranteed

$$\omega_k = \frac{k_y u_d}{1+k_\perp^2}$$ difference between x and y direction

  consequently

$$W(\mathbf{k}) = W(k_x, k_y) = \frac{c}{k_x^\alpha k_y^\beta}$$ with $\alpha \neq \beta$ as possible solution

- The above introduces two dimensionless quantities $k_s\rho_s$ and (k_x/k_y)
  Pure dimensional arguments are incomplete.

Kolmogorov spectrum for isotropic Hasegawa-Mima

For k>>1, the Hasegawa-Mima model is the same as the 2D incompressible fluid, and the spectrum is the classical Kolmogorov one.

For k<<1 the Hasegawa-Mima equation becomes $\frac{\partial}{\partial t}\phi - (\mathbf{u}_E \cdot \nabla)\nabla^2\phi = 0$
with energy and enstrophy

$$W = \int dV \phi^2 \qquad U = \int dV \nabla^2 \phi$$

Now the total energy W(k)dk is not analogous to u_k^2.

Scale transformation: (x,y)→λ(x,y), t→τt leaves HW invariant if $\phi \to \frac{\lambda^4}{\tau}\phi$ or $\phi \sim \frac{\lambda^4}{\tau}$

Then the energy transfer is $\epsilon = const = \frac{\phi^2}{\tau} \sim \frac{\lambda}{\tau^3} \Rightarrow \tau \sim \epsilon^{-1/3} k^{-8/3}$

And the energy spectrum is $W_k = \frac{\phi^2}{k} \sim \frac{\lambda^8}{\tau^2 k} \sim \epsilon^{2/3} k^{-11/3}$ inverse cascade

If we do the same calculation for the enstrophy, the spectrum is $W_k \sim \eta^{2/3} k^{-5}$
direct cascade

Energy spectrum: Anisotropy

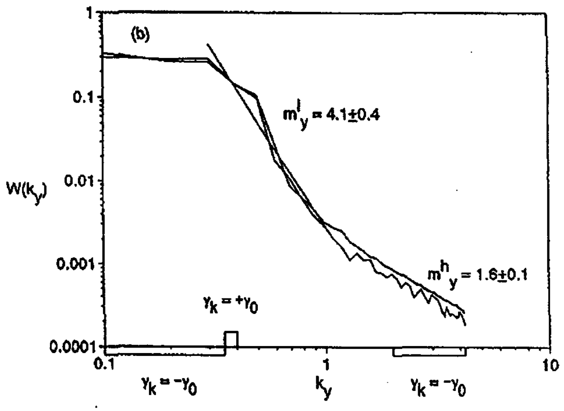
If we take into account the anisotropy, then

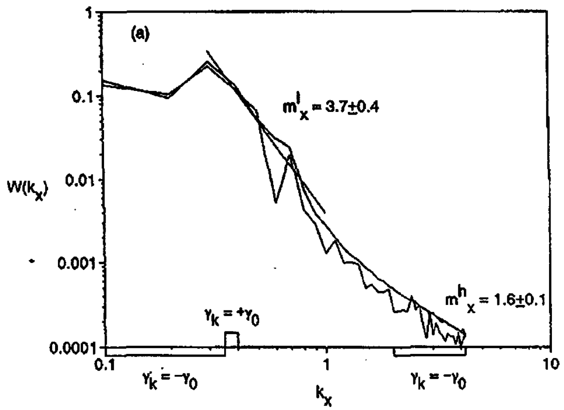
$$W \propto k_x^{-\alpha} k_y^{-\beta} \qquad \alpha \neq \beta$$

Wave coupling theory gives

$$W(k_x, k_y) = \begin{cases} k_x^{-2} k_y^{-3/2} & k\rho_s >> 1 \\ \\ k_x^{-3} k_y^{-3/2} & k\rho_s << 1 \end{cases}$$

One dimentional energy spectra. Exponents do not fit exactly with analytic results.





Coherent structures

Also in Hasegawa-Mima there exist structures that are long lived (longer than usual with turn over time t ≠ 1 /u k)

Single vortex solutions

center of the vortex

$$\phi = \omega K_0(|\mathbf{r} - \mathbf{r}_c|)$$

Modified Bessel function of second kind

compared with the 2D Euler's vortex solution

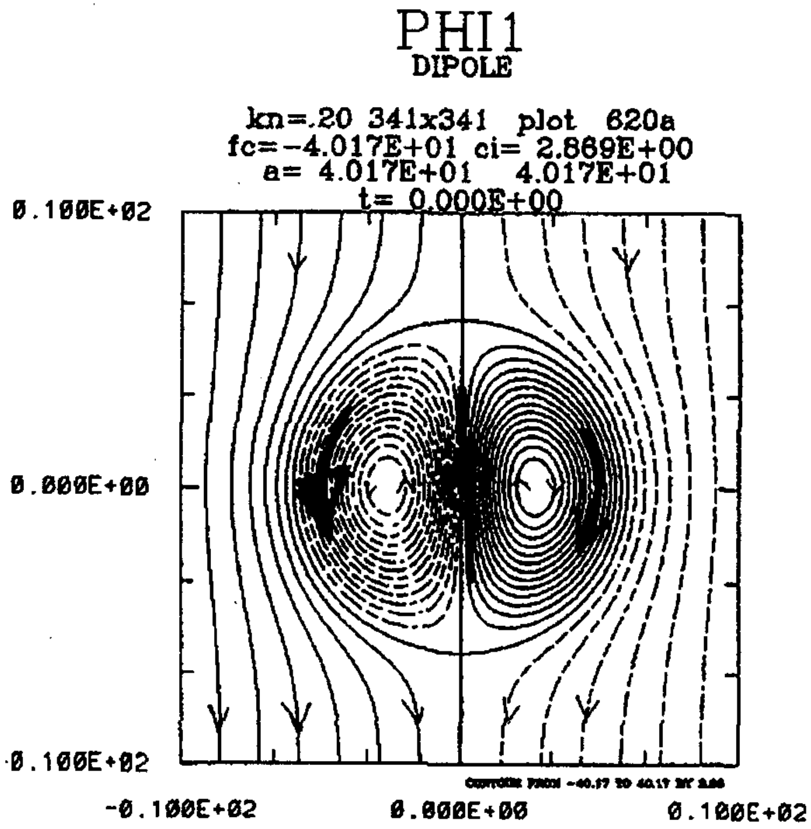
$$\phi = \omega ln(|\mathbf{r} - \mathbf{r}_c|)$$

The solution of the Hasegawa-Mima equation dies off exponentially for $|\mathbf{r} - \mathbf{r}_c| >> 1$ (shielded vortex), i.e. they have a finite size.

Dipole vortex

Hasegawa-Mima also admits a stationary vortex pair solution.

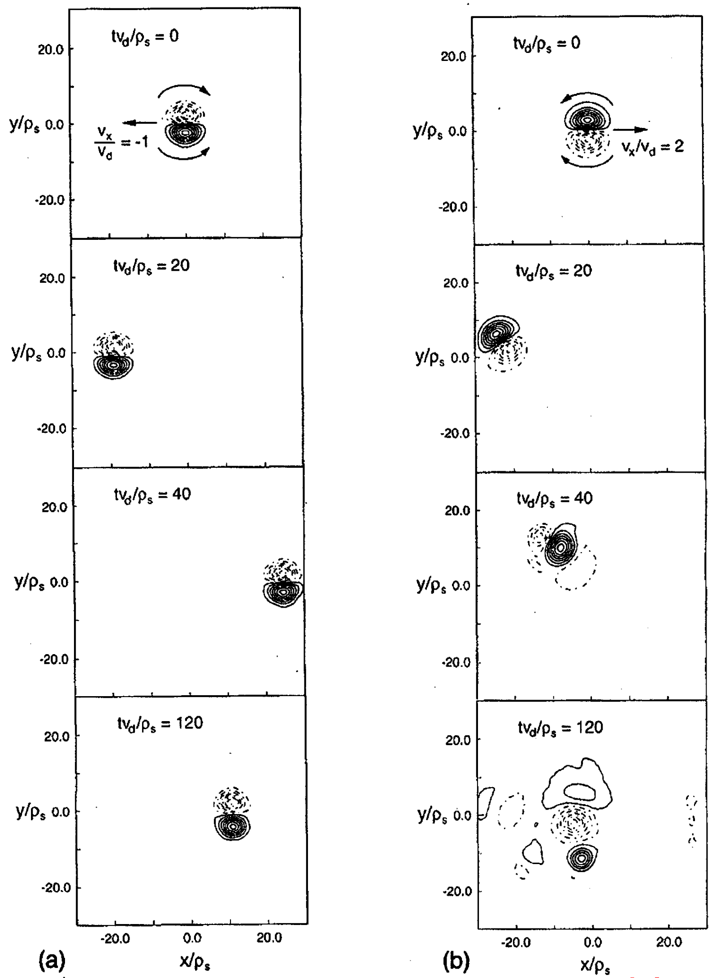
The dipole vortices live long and have some robustness against perturbations. (eg. collisions)



Cross-sectional profiles

Coherent structures

Dipole structure with small perturbation



stable unstable

- Much of the behaviour is similar to the 2D fluid, and has been shown in the previous talk. For instance, vortex merging occurs as in clear from the movie.

- Differences can occur. For instance the anisotropy makes that dipoles propagating in opposite direction (ion or electron diamagnetic) have different stability properties.

The Hasegawa-Wakatani model

The Hasegawa-Wakatani model

- Consistent description of drift wave turbulence, no forcing added by hand.
- Includes two variables, density and electric potential.
- As in Hasegawa-Mima, homogeneous magnetic field ($\mathbf{B}=B_o\,\mathbf{z}$) and inhomogeneous plasma ($n_o = n_o(\mathbf{x})$)
- Electron-ion collisions and ion viscosity are kept.

Derivation of the Hasegawa-Wakatani equations

- Start again with ion momentum balance

$$\frac{d\mathbf{u}}{dt} = -\frac{e}{m}\nabla\phi + \frac{e}{m}\mathbf{u}\times\mathbf{B} - \nabla p_i - \nabla\Pi + \mathbf{F}$$

and continuity

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0$$

- Now we assume

$$\nabla p_i = 0 \quad \mathbf{F} = 0 \quad -\nabla\Pi = \mu\nabla^2\mathbf{u}$$ viscosity, where μ is the ion viscosity coefficient

Again

$$\mathbf{u}^0 = \mathbf{u}_\mathrm{E} \quad \mathbf{u}^1 = -\frac{1}{\omega_{ci}B_0}\frac{d\nabla\phi}{dt} - \frac{\mu}{\omega_{ci}B_0}\mathbf{B}\times\nabla^2\mathbf{u}^0$$

polarisation as before

viscosity Here put $\mathbf{u}^0 = \mathbf{u}_\mathrm{E}$

$$\mathbf{u}^1_{visc} = \frac{\mu}{\omega_{ci}B_0}\nabla^2(\nabla\phi)$$ ← additional velocity (non divergence free)

Derivation of the Hasegawa-Wakatani equations

Substituting the total velocity in the continuity equation, one has

$$\frac{d}{dt}\left(\frac{\nabla^2\phi}{\omega_{ci}B_0} - \frac{n}{n_0} - ln n_0\right) - \mu\frac{\nabla^4\phi}{\omega_{ci}B_0} = 0$$

additional term

⇨ For the electrons, we start again with the momentum balance

$$\frac{d\mathbf{u}}{dt} = -\frac{e}{m}\nabla\phi + \frac{e}{m}\mathbf{u}\times\mathbf{B} - \nabla p - \nabla\Pi + \mathbf{F}$$

where we put

$$m_e n_e \frac{d\mathbf{u}}{dt} = 0 \quad \nabla\Pi = 0$$

but we keep the electron-ion friction

$$F_{e||} = -m_e n_e \nu_{ei} u_{e||} = \frac{m_e \nu_{ei}}{e} J_{||}$$

Thus for the parallel dynamics

$$\nabla_{||}\phi + \eta J_{||} = \frac{1}{en_0}\nabla_{||}p_e \Rightarrow J_{||} = \frac{T_e}{e\eta}\nabla_{||}\left(\frac{n}{n_0} - \frac{e\phi}{T_e}\right)$$

where η: resistivity

Derivation of the Hasegawa-Wakatani equations

For the electrons the polarisation drift is ignored (mass ratio). Then the continuity becomes

$$\frac{\partial n}{\partial t} + \mathbf{u}_E \cdot \nabla n + n\nabla_{||} u_{e||} = 0 \Rightarrow \frac{d}{dt}\left(\frac{n}{n_0} + ln n_0\right) = \frac{1}{en_0}\nabla_{||} J_{||}$$

where $\nabla \mathbf{u}_E = 0$ and $v_{i//}=0$

One can now eliminate J to get

$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right) \nabla^2 \phi = c_1(\phi - n) + c_2 \nabla^4 \phi$$

$$\left(\frac{\partial}{\partial t} - \nabla\phi \times \hat{z} \cdot \nabla\right) (n + ln n_0) = c_1(\phi - n)$$

Hasegawa-Wakatani equations

where $c_1 = -\frac{T_e}{e^2 n_0 \eta \omega_{ci}} \nabla_{||}^2 \quad c_2 = \frac{\mu}{\rho_s^2 \omega_{ci}}$

- Hasegawa-Mima limit: $c_1 >> 1$ and $c_2 << 1$ (collisionless plasma)
- Euler limit: $c_1 << 1$ and $c_2 << 1$ (incompressible fluid)

Dispersion relation for Hasegawa-Wakatani

$$\phi \rightarrow \phi_{\mathbf{k}} exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) \quad n \rightarrow n_{\mathbf{k}} exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

yields two algebraic coupled equations

$$i\omega k^2 \phi_{\mathbf{k}} = c_1(\phi_{\mathbf{k}} - n_{\mathbf{k}}) + c_2 k^4 \phi_{\mathbf{k}}$$

$$-i\omega n_{\mathbf{k}} - i(\mathbf{k}\times\hat{z})\cdot\nabla ln n_0 \phi_{\mathbf{k}} = c_1(\phi_{\mathbf{k}} - n_{\mathbf{k}})$$

The nontrivial solution then gives the dispersion relation

$$\omega\left(1 + \frac{k^2}{1 - i\omega/c_1}\right) = -(\mathbf{k}\times\hat{z})\cdot\nabla ln n_0 - i c_2 k^4$$

small correction for $c_1 >> \omega$

Negative imaginary contribution to the frequency.
Leads to a damping of the wave increasing strongly with k

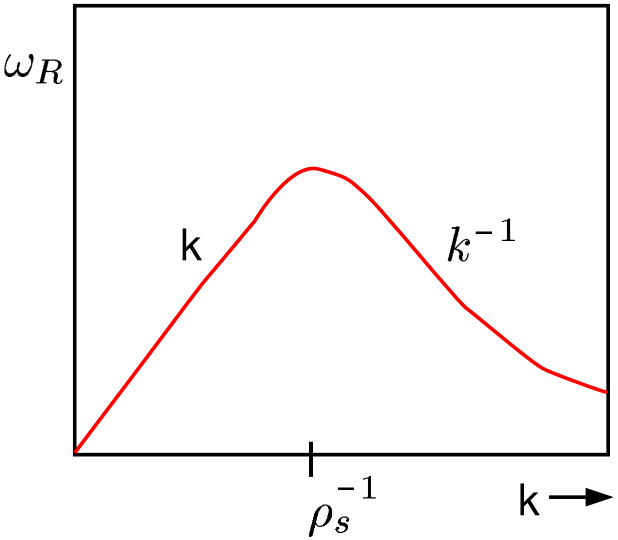
Dispersion relation

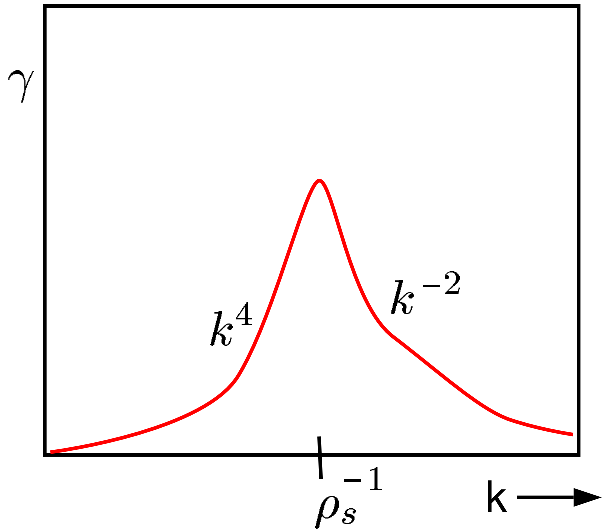
$$\omega\left(1+\frac{k^2}{1-i\omega/c_1}\right) = -(\mathbf{k}\times\hat{z})\cdot\nabla ln n_0 - ic_2 k^4$$

With $c_1 >> \omega \quad and \quad c_2 << \omega$ one obtains the solution

$$\omega = \omega_R + i\gamma \quad with \quad \gamma << \omega_R$$

$$\gamma = \frac{k^2\omega_R^2}{1+k^2}\frac{1}{c_1} - c_2\frac{k^4}{1+k^2} \qquad \omega_R = \frac{k_y v_d}{1+k^2}$$





The electron-ion friction leads to an instability with a growth rate strongly peaked around

$$k\rho_s = 1$$

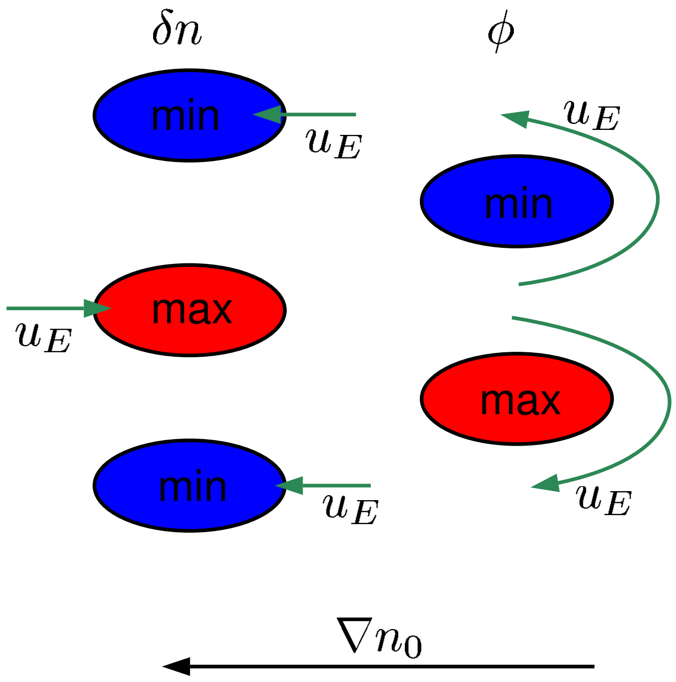
Unstable solutions

$$J_{\parallel} = \frac{T_e}{e\eta} \nabla_{\parallel} \left(\frac{n}{n_0} - \frac{e\phi}{T_e} \right)$$

For finite friction, η is finite and adiabaticity is not exactly satisfied.

$$\frac{n}{n_0} = \frac{e\phi}{T_e} + i\delta$$

This phase shift makes that particles from the left (where the density is higher) move into the maximum of the density perturbation enhancing the perturbation (unstable solution)



Energy evolution

$$\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\int dV [n^2 + (\nabla\phi)^2] =$$

$$-\frac{c_1}{k_z^2}\int dV \left(\frac{\partial n}{\partial z} - \frac{\partial \phi}{\partial z}\right)^2 - c_c \int dV (\nabla^2\phi)^2 - \int dV n(\hat{z}\times\nabla ln n_0)\cdot\nabla\phi$$

always negative

can be positive

$$n(\hat{z}\times\nabla ln n_0)\cdot\nabla\phi = n(\nabla\phi\times\hat{z})\cdot\nabla ln n_0 =$$

$$= -n\,\mathbf{u}_E\cdot\nabla ln n_0$$ particle flux

The growth of the energy is related to the existence of a net particle flux.

In reality…

Hasegawa-Wakatani is only suited to study basic physics. In reality

- ➪ Temperature perturbations can not be neglected
- ➪ Parallel dynamics play a role
- ➪ The magnetic field is inhomogeneous
- ➪ The perturbations are not necessarily electrostatic.

Summary

Hasegawa-Mima model

⇛ 2D equation for potential. No source or sink
⇛ Adiabatic electrons
⇛ Similar to 2D incompressible fluid
⇛ Generalised energy and enstrophy conserved. 2D Kolmogorov-like spectrum.
⇛ Description of drift wave, but not instability

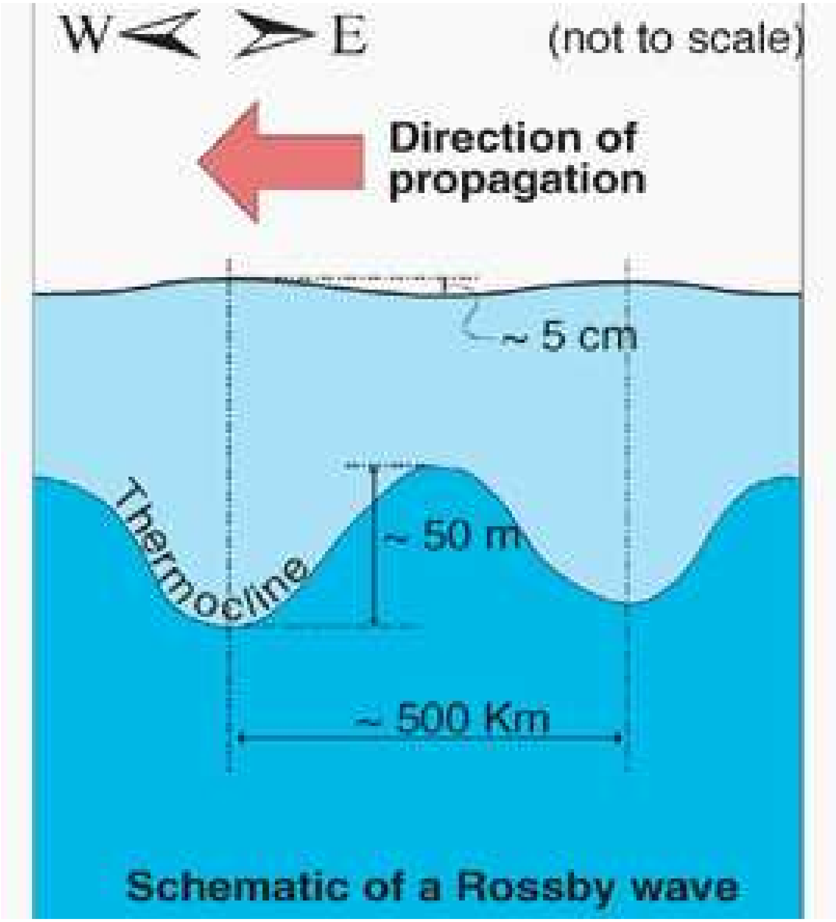
Hasegawa-Wakatani model

⇛ 2D equations for potential and density
⇛ Electrons non-adiabatic. Collisions and viscosity kept.
⇛ Instability accurs, growth rate peaks at $k\rho_s = 1$

Connection with Geophysics

Rossby waves

- Created due to planetary rotation
- On Earth seen in atmosphere and oceans
- Difficult to observe at oceans ⇛ big difference in horizontal and vertical scale lengths. Measured by satelites
- Always travel from East to West following the parallels.
- Speed a few cm/s
  takes several months to cross Pacific ocean
- Have major effects on ocean circulation (weather and climate!)
  Can change currents



Physical picture of Rossby wave: Thin fluid layer

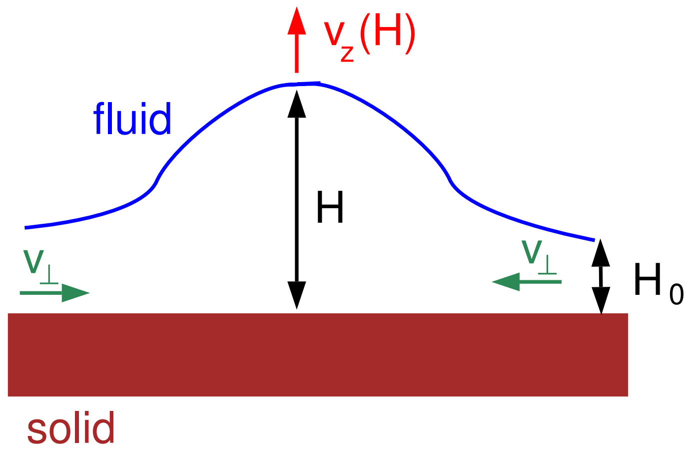
- Fluid is incompressible, but has a free surface
- Fluid can escape in the third dimension

$$\nabla \cdot \mathbf{u} = 0 \quad but \quad \nabla \cdot \mathbf{u}_\perp = -\nabla \cdot \mathbf{u}_{||}$$

- Use the 2D approximation

$$\mathbf{u}_\perp = \mathbf{u}_\perp(x,y), \quad H = H(x,y), ...$$

Thin fluid layer with a free surface



$$\frac{dH}{dt} = v_z(H) = \int_0^H dz \frac{\partial u_z}{\partial z} = - \int_0^H dz \nabla \cdot \mathbf{u}_\perp = -H \nabla \cdot \mathbf{u}_\perp$$

$u_z(0) = 0$ $\qquad$ $\nabla \cdot \mathbf{u} = 0$

Physical picture of Rossby wave: Pressure of the fluid

The motion of the fluid

$$mn\frac{d\mathbf{u}}{dt} = -\nabla p$$

is influenced through the change of the pressure

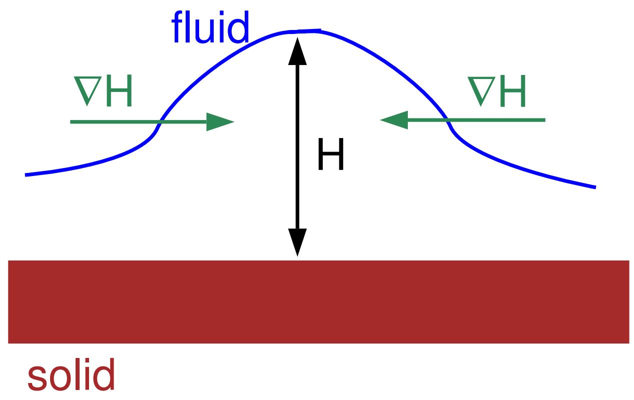
$$p = mngH$$

Therefore

$$\frac{d\mathbf{u}}{dt} = -g\nabla H$$

Note

$$H = H(x,y) \qquad \nabla H \perp \mathbf{u}_z$$



⇛ Fluid wants to flow such that H becomes uniform.

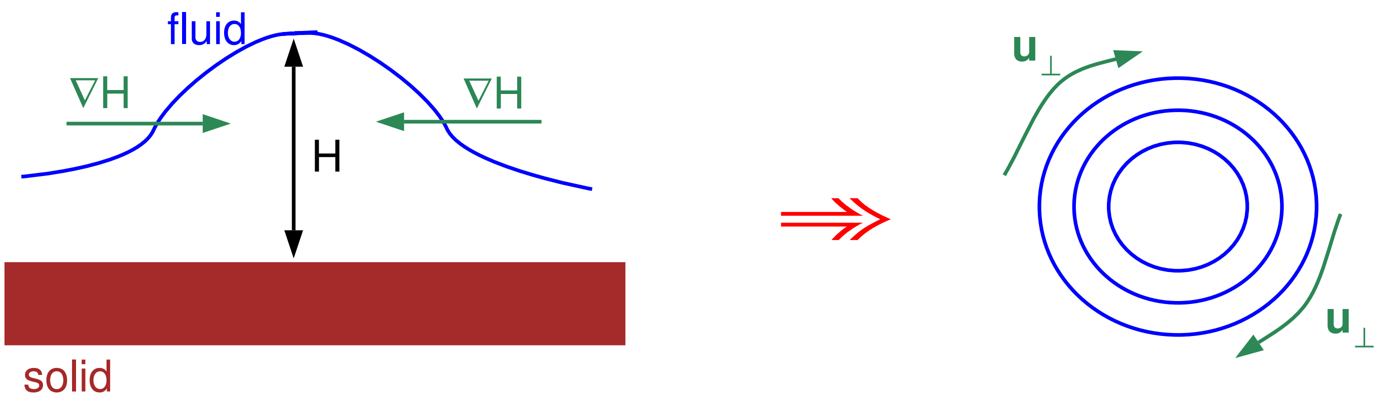
Physical picture of Rossby wave: Coriolis force

- Besides the pressure gradient, also the Coriolis force is considered

$$\frac{d\mathbf{u}}{dt} = -g\nabla H + f\mathbf{u} \times \hat{z}$$

This force plays the role of the magetic field. If it is strong

$$-g\nabla H = f\mathbf{u} \times \hat{z} \Rightarrow \mathbf{u}_\perp = -\frac{g}{f}\hat{z} \times \nabla H$$



- The fluid moves along contours of constant H, known as Geostrophic flow (ExB velocity of drift waves).

Charney model

- Start with 2D velocity of atmospheric flow in horizontal plane

$$\frac{d\mathbf{v}}{dt} = -g\nabla H + f\mathbf{v} \times \hat{z}$$

Coriolis force with Coriolis parameter f

Horizontal acceleration of a constant mass density fluid

gradient of hydrostatic pressure p=ρgH where H is the total depth of the atmosphere

- The continuity equation can be written as

$$\nabla \cdot \mathbf{v} = -\frac{d}{dt} lnH \qquad \text{where} \quad H = H_0 + h \quad h << H_0$$

- Introduce a small parameter and normalisation

$$\epsilon = \frac{1}{f}\frac{\partial}{\partial t}$$

and do the same derivation as before.

Charney model

- Introduce normalisation

$$ft \to t \quad \frac{x,y}{\rho_g} \to x,y \quad \frac{h}{H_0} \to h \quad \text{where} \quad \rho_g = \frac{(gH_0)^{1/2}}{f}$$

- To get

$$\frac{\partial}{\partial t}(\nabla^2 h - h) - [(\nabla h \times \hat{z}) \cdot \nabla] \left(\nabla^2 h - ln \frac{H_0}{f} \right) = 0$$

Charney equation

- Charney's equation gives the dispersion relation for the Rossby waves

$$\omega_k = \frac{(\mathbf{k} \times \hat{z}) \cdot \nabla ln f}{1 + k^2}$$

similar to that of the drift waves

Drift wave versus Rossby

The equations for the evolution are equivalent. Only the symbols need to be differently interpreted.

Electrostatic potential $\phi(x, y)$ $\Rrightarrow$ Variable fluid depth $h(x, y)$

Lorentz force $m\omega_{ci}u_\perp$ $\Rrightarrow$ Coriolis force

Cyclotron frequency ω_{ci} $\Rrightarrow$ Coriolis parameter f

Ion acoustic speed c_s $\Rrightarrow$ Gravity wave speed $(gH)^{1/2}$

ExB velocity $\Rrightarrow$ Geostrophic flow $\mathbf{u}_\perp = \left(\frac{g}{f}\right) \hat{z} \times \nabla h$