Theory and Computation in Full-F Gyrokinetics

B. Scott

Max Planck Institut für Plasmaphysik
Euratom Association
D-85748 Garching, Germany

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Outline

- **Preliminaries**

  — the edge situation

  — basic ideas for fully consistent gyrokinetics

- **What Gyrokinetics Means, Gyrokinetic Equation**

  — electromagnetic, fully nonlinear

  — Lagrangian, Hamiltonian structure

  — variational principle → polarisation

- **Energy Conservation**

  — particle versus field Hamiltonian

  — drift energy versus thermal energy

  — polarisation equation and energy theorem

  — phase space conservation

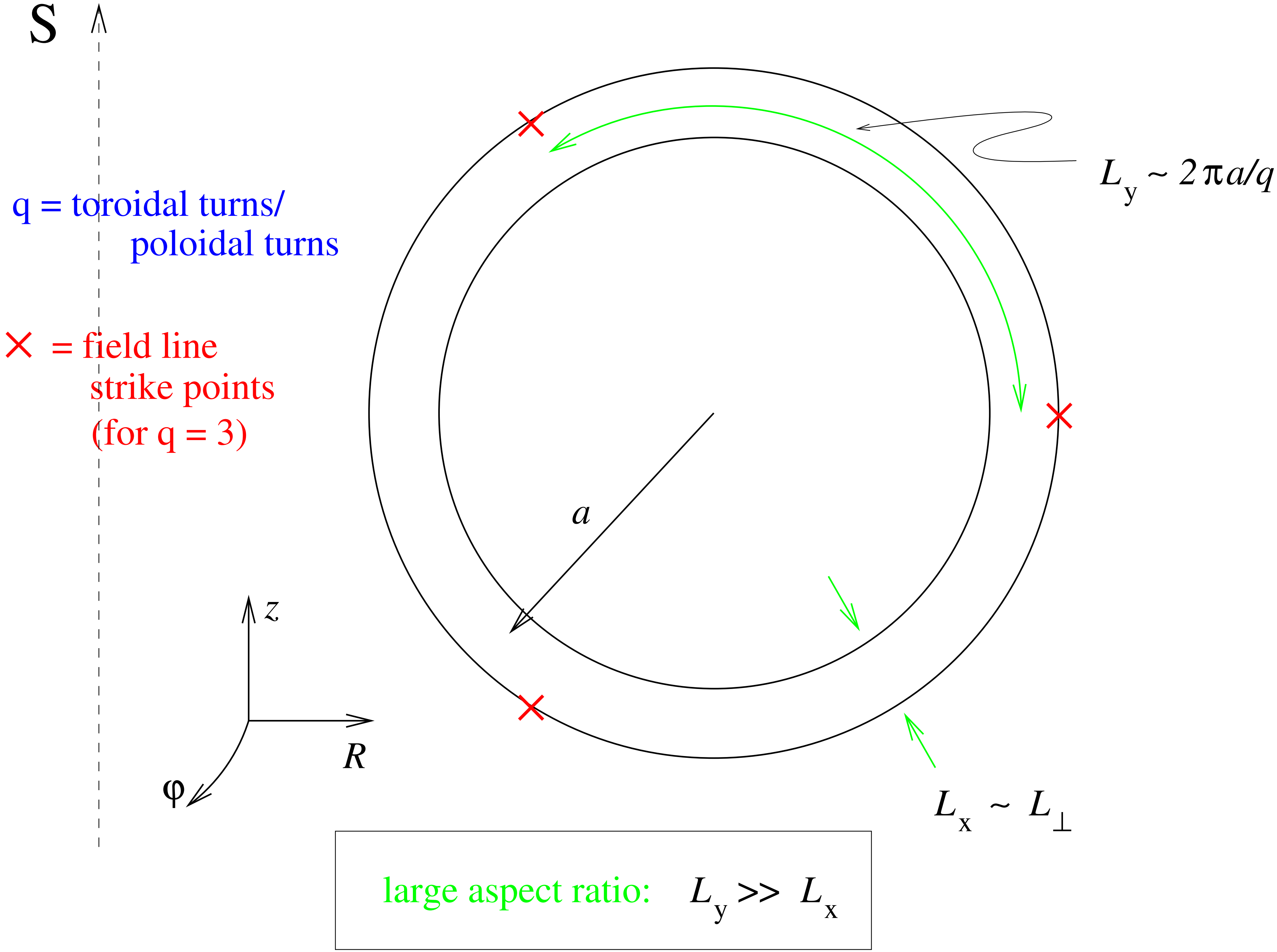
- **The FEFI Code**

  — implementation of above ideas

  — energy conservation, collision model

  — kinetic shear Alfvén results

Computational Domain



edge parameters

- typical situation: Alfvén/electron transit, collision, and drift frequencies comparable
- drift frequency is $c_s/L_\perp$, spectral range of main interest is $0.1 < k_y\rho_s < 1$
- steep gradient

$$\hat{\mu} \equiv \frac{m_e}{M_D}\left(\frac{qR}{L_\perp}\right)^2 = \left(\frac{c_s/L_\perp}{V_e/qR}\right)^2 > 1$$

- collisional

$$C \equiv \frac{0.51\nu_e}{c_s/L_\perp}\frac{m_e}{M_D}\left(\frac{qR}{L_\perp}\right)^2 = 0.51\frac{\nu_e\, c_s/L_\perp}{(V_e/qR)^2} > 1$$

- electromagnetic

$$\hat{\beta} \equiv \frac{4\pi p_e}{B^2}\left(\frac{qR}{L_\perp}\right)^2 = \left(\frac{c_s/L_\perp}{v_A/qR}\right)^2 \gtrsim 1$$

Basic Ideas

- particle Lagrangian L_p
- continuum: distribution function times L_p, plus field contributions
- incorporate fields $\rightarrow$ full self consistency
- Noether theorem $\rightarrow$ conservation laws for *this* system

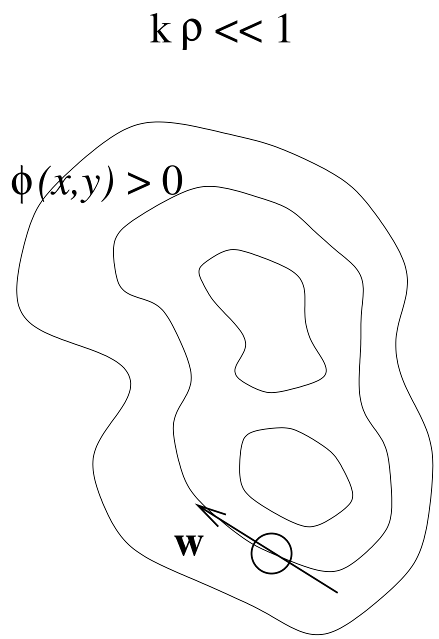
Why Continuum?

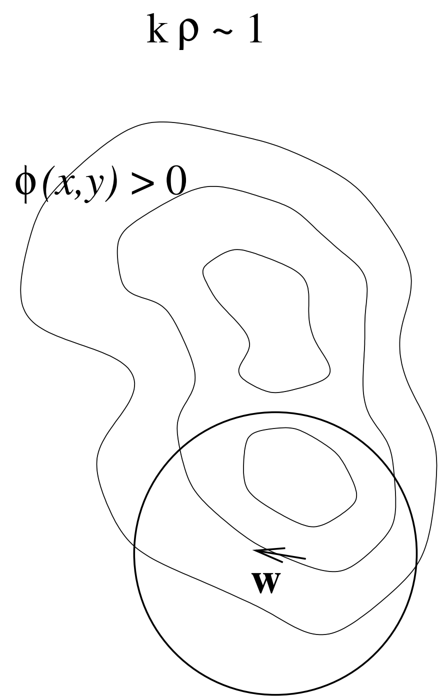
- “Vlasov model”
- usual blah blah about noise not the real reason …
- with 20 years fluid experience, it is what I know
  - ∘ and is now becoming tractable
- same basic methods as mature GEM (gyrofluid electromagnetic) code

The Meaning of Gyrokinetics

- low frequencies $\omega \ll \Omega_c = eB/mc$ for each species

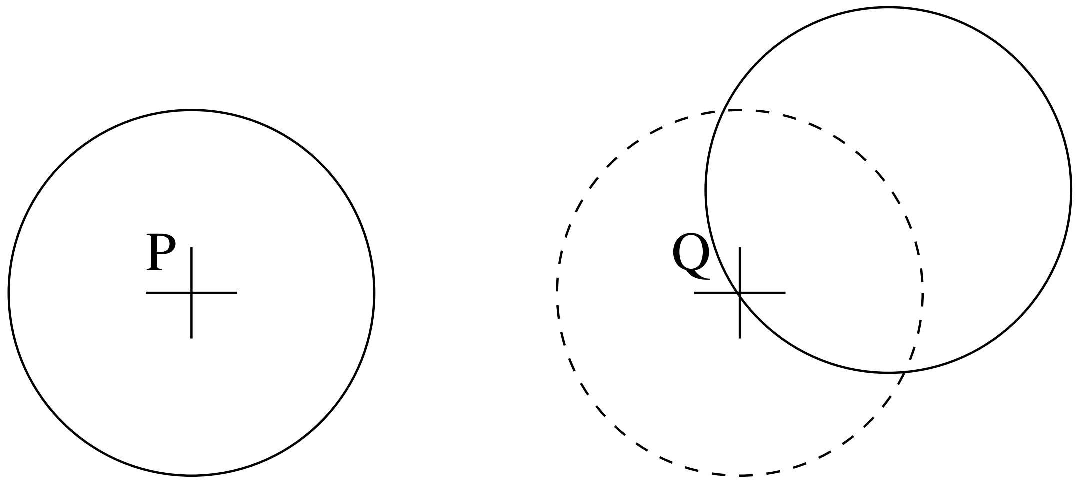
treat particles as rings of charge in spatially varying fields





- reduced response: "gyroaveraging"
- reaction to fields, polarisation density: "gyroscreening"

Gyrokinetic/Gyrofluid Polarisation



gyrocenter at P senses potential at points on the ring

gyrocenters on dashed ring contribute charge to Q

mathematics identical in both procedures (Hermitian operators)

polarisation density <–– part of f dependent upon gyrophase angle

Basic Procedure

Guiding Center Coordinates

- phase space extention of original EM Lagrangian → “fundamental one form”

$$L = m\frac{|\dot{\mathbf{x}}|^2}{2} + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{x}} - e\phi$$

  becomes

$$\gamma = \left(\frac{e}{c}\mathbf{A} + m\mathbf{v}\right)\cdot\mathbf{dx} - H\,dt$$

  with

$$H = m\frac{v^2}{2} + e\phi$$

- change to guiding center coordinates for motion in background magnetic field

$$\gamma(\mathbf{X}, U, \mu, \theta) = \left(\frac{e}{c}\mathbf{A} + mU\mathbf{b}\right)\cdot\mathbf{dR} + \mu B\,\frac{d\theta}{\Omega} - H_0\,dt$$

  where θ is gyrophase angle, $\mathbf{R} = \mathbf{x} - \rho_L\mathbf{a}$ is guiding center position, and

$$H_0 = m\frac{U^2}{2} + \mu B \qquad \mu = \frac{mv_\perp^2}{2B} \qquad U = v_\parallel$$

- ϕ and $A_\parallel$ are the dynamical fields to be added later

Lie Transform to Gyrocenter Coordinates

- **Hahm 1988:** Lie transform, counts fields as potential energy, part of Hamiltonian

$$\Gamma_{(0,1)} = \left[\frac{e}{c}\left(\mathbf{A} + \left\langle A_\parallel \right\rangle \mathbf{b}\right) + mU\mathbf{b}\right] \cdot \mathbf{dR} + \mu B \,\frac{d\theta}{\Omega} - H_{(0,1)}\, dt$$

where

$$H_{(0,1)} = m\frac{U^2}{2} + \mu B + e\left\langle\phi\right\rangle + \frac{e^2}{B}\frac{\partial}{\partial\mu}\left(\phi - \left\langle\phi\right\rangle\right)^2$$

- Euler Lagrange equations for gyrocenters

$$\dot{\mathbf{R}} = \mathbf{b}^*\left(\frac{1}{m}\frac{\partial H}{\partial U}\right) - \frac{c}{eBB_\parallel^*}\mathbf{F}\cdot\nabla H \qquad \dot{U} = -\frac{1}{m}\mathbf{b}^*\cdot\nabla H - \frac{e}{mc}\frac{\partial\left\langle A_\parallel\right\rangle}{\partial t}$$

where

$$\mathbf{B}^* = \nabla\times\left[\left(\mathbf{A} + \left\langle A_\parallel\right\rangle\mathbf{b}\right) + \frac{mc}{e}U\mathbf{b}\right] \qquad B_\parallel^* = \mathbf{b}\cdot\mathbf{B}^* \qquad \mathbf{b}^* = \mathbf{B}^*/B_\parallel^*$$

with gyromotion

$$\dot{\theta} = \frac{\Omega}{B}\frac{\partial H}{\partial\mu} = \Omega \qquad \dot{\mu} = -\frac{\Omega}{B}\frac{\partial H}{\partial\theta} = 0$$

make approximations in Lagrangian
rest is rigorously consistent

The Gyrokinetic Model

- electromagnetic, full-f
- particle Lagrangian

$$L_p = \left[\frac{e}{c}\left(\mathbf{A} + A_\parallel \mathbf{b}\right) + m v_\parallel \mathbf{b}\right] \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\dot{\theta} - \left(m\frac{v_\parallel^2}{2} + \mu B + e\psi\right)$$

- Hamiltonian

$$H = m\frac{v_\parallel^2}{2} + \mu B - e\psi$$

- gyroaveraged and (low-$k_\perp$) screened electrostatic potential

$$e\psi = eJ_0\phi - m\frac{v_E^2}{2}$$

- modified field

$$\mathbf{A}^* = \mathbf{A} + A_\parallel \mathbf{b} + \frac{mc}{e} v_\parallel \mathbf{b} \qquad \mathbf{B}^* = \nabla\times\mathbf{A}^* \qquad B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b} \qquad \mathbf{b}^* = \frac{\mathbf{B}^*}{B_\parallel^*}$$

Some Notes

- **computational approximations:** change Lagrangian ...

$$L_{(0,1)} = \left[\frac{e}{c}\left(\mathbf{A} + \langle A_\parallel \rangle \mathbf{b}\right) + mU\mathbf{b}\right] \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\dot{\theta} - H_{(0,1)}$$

  ... to this version:

$$L_p = \left[\frac{e}{c}\left(\mathbf{A} + A_\parallel \mathbf{b}\right) + mv_\parallel \mathbf{b}\right] \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\dot{\theta} - H$$

  and change Hamiltonian ...

$$H_{(0,1)} = m\frac{U^2}{2} + \mu B + e\langle\phi\rangle + \frac{e^2}{B}\frac{\partial}{\partial\mu}\left(\phi - \langle\phi\rangle\right)^2$$

  ... to this version:

$$H = m\frac{v_\parallel^2}{2} + \mu B + eJ_0\phi - m\frac{v_E^2}{2}$$

- $A_\parallel$ is not gyroaveraged: belongs to electrons
- range of scales: profile scale $L_\perp$ down to drift scale ρ_s, comparable to ion gyroradius ($T_i \sim T_e$)
- low-$k_\perp$ form for screening: important mostly at large scale, where strongest velocities are
- forms motivated by computational tractability

One More Note

- **correspondence to Hahm 1996:** in electrostatic limit change Lagrangian . . .

$$L_\phi = \left[\frac{e}{c}\mathbf{A} + mU\mathbf{b} + m\mathbf{v}_E\right] \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\,\dot{\theta} - eJ_0\phi - m\frac{v_E^2}{2}$$

. . . to this version:

$$L_p = \left[\frac{e}{c}\mathbf{A} + mv_\parallel\mathbf{b}\right] \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\dot{\theta} - eJ_0\phi + m\frac{v_E^2}{2}$$

the 1996 version Part I is valid for arbitrary ϕ-amplitude
hence L_p is valid for any ϕ but small $k_\perp^2\rho_L^2$
or for small ϕ but any $k_\perp^2\rho_L^2$ (up to the validity of J_0 or polarisation)

- changing $m\mathbf{v}_E \cdot \dot{\mathbf{R}} - mv_E^2/2$ to $mv_E^2/2$ through a Lie transform (simple Lie derivative)
- note in energy: under phase space $\int d\Lambda$ we will see that $f\,eJ_0\phi$ is equivalent to $f\,mv_E^2$
- note in frequency ordering: $k_\perp^2\rho_L^2(e\phi/T) \ll 1$ is the same as vorticity $\ll$ gyrofrequency

Field Theory Description

- combine particles/fields into a **system Lagrangian**

$$L = \sum_{\text{sp}} \int d\Lambda\, f\,[L_p] - \int dV\, \frac{B_\perp^2}{8\pi}$$

where

- $d\Lambda = dW\,dV$ is the phase space element, with velocity-space and volume elements dW and dV
- magnetic disturbances $\mathbf{B}_\perp$ with magnitude $B_\perp$ given by

$$\mathbf{B}_\perp = \nabla\times(A_\parallel \mathbf{b}) \qquad B_\perp^2 = \left|\nabla_\perp A_\parallel\right|^2$$

ultimately ...

- equations of motion arise from varying particle coordinates
- and field equations arise from varying field potentials
- and energy theorem arises from application of the Noether theorem

basics from Sugama (PoP 2000) and Brizard (PoP 2000)

Particle Lagrangian

- repeated, to recall the definitions of $\mathbf{A}^*$ and H etc.
- particle Lagrangian

$$L_p = \left[\frac{e}{c}\left(\mathbf{A} + A_\parallel \mathbf{b}\right) + m v_\parallel \mathbf{b}\right] \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\dot{\theta} - \left(m\frac{v_\parallel^2}{2} + \mu B + e\psi\right)$$

- Hamiltonian

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- gyroaveraged and (low-$k_\perp$) screened electrostatic potential

$$e\psi = eJ_0\phi - m\frac{v_E^2}{2}$$

- modified field

$$\mathbf{A}^* = \mathbf{A} + A_\parallel \mathbf{b} + \frac{mc}{e} v_\parallel \mathbf{b} \qquad \mathbf{B}^* = \nabla\times\mathbf{A}^* \qquad B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b} \qquad \mathbf{b}^* = \frac{\mathbf{B}^*}{B_\parallel^*}$$

The Gyrokinetic Equation

- particle Lagrangian

$$L_p = \frac{e}{c}\mathbf{A}^* \cdot \dot{\mathbf{R}} + \frac{\mu B}{\Omega}\dot{\theta} - H$$

- equation of motion for distribution function

$$\frac{\partial f}{\partial t} - \frac{e}{m}\frac{\partial f}{\partial v_\parallel}\,\frac{1}{c}\frac{\partial A_\parallel}{\partial t} = \frac{1}{m}\left(\frac{\partial f}{\partial v_\parallel}\mathbf{b}^* \cdot \nabla H - \frac{\partial H}{\partial v_\parallel}\mathbf{b}^* \cdot \nabla f\right) + \frac{c}{eBB_\parallel^*}\mathbf{F}:(\nabla H \nabla f)$$

- tensor representation for magnetic field

$$\mathbf{F} = \epsilon \cdot \mathbf{B} \qquad \text{hence} \qquad \mathbf{v}_E = \frac{c}{B^2}\mathbf{B}\times\nabla\phi = -\frac{c}{B^2}\mathbf{F}\cdot\nabla\phi$$

- Hamiltonian structure

$$\frac{\partial f}{\partial t} - \frac{e}{m}\frac{\partial f}{\partial v_\parallel}\,\frac{1}{c}\frac{\partial A_\parallel}{\partial t} = [H, f]$$

where, as we will see, the term $\partial A_\parallel/\partial t$ tracks magnetic energy,

and where ...

... where we have a spatial coordinate system $\{x, y, s\}$ so that

$$[H, f] = \frac{b^{*x}}{m}\left(\frac{\partial f}{\partial v_\parallel}\frac{\partial H}{\partial x} - \frac{\partial H}{\partial v_\parallel}\frac{\partial f}{\partial x}\right) + \frac{c\,F^{xy}}{eBB^*_\parallel}\left(\frac{\partial f}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial y}\right)$$

$$+\frac{b^{*y}}{m}\left(\frac{\partial f}{\partial v_\parallel}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial v_\parallel}\frac{\partial f}{\partial y}\right) + \frac{c\,F^{xs}}{eBB^*_\parallel}\left(\frac{\partial f}{\partial x}\frac{\partial H}{\partial s} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial s}\right)$$

$$+\frac{b^{*s}}{m}\left(\frac{\partial f}{\partial v_\parallel}\frac{\partial H}{\partial s} - \frac{\partial H}{\partial v_\parallel}\frac{\partial f}{\partial s}\right) + \frac{c\,F^{ys}}{eBB^*_\parallel}\left(\frac{\partial f}{\partial y}\frac{\partial H}{\partial s} - \frac{\partial H}{\partial y}\frac{\partial f}{\partial s}\right)$$

- notes:
  - ∘ curvature drift is in the b^* terms
  - ∘ grad-B drift is in the μB part of the drift terms
  - ∘ (generalised) ExB drift is in the $e\psi$ part of the drift terms
  - ∘ parallel dynamics is in the b terms

- note especially:
  - ∘ parallel dynamics and trapping go together, in both $e\psi$ and μB
  - ∘ (really subtle) phase space conservation:

$$\frac{e}{m}\frac{\partial}{\partial v_\parallel}\left(\mathbf{B}^*\cdot\nabla\psi\right) \qquad \text{versus} \qquad \nabla\cdot\frac{c}{B}\mathbf{F}\cdot\nabla\psi$$

∘ that is, curvature drift part of trapping versus ExB advection!

Polarisation: Reaction of the Field

- find from Hamilton's principle, variation of ϕ

$$\sum_{\rm sp}\left[\int dW\ eJ_0 f + \nabla\cdot\frac{nmc^2}{B^2}\nabla_\perp\phi\right] = 0 \qquad \text{where} \qquad n = \int dW\ f$$

- a little slower ...
  - position gyrocenters in space (G is positioning kernel), don't forget sum over species:

$$\int dV\,\mathcal{L} = \sum_{\rm sp}\int dV\ \sum_p L_p\,G(\mathbf{R},\mathbf{x})$$

  - examine part due to the potential ϕ:

$$\int dV\,\mathcal{L} = \sum_{\rm sp}\int dV\ \int dW\ f\,(\cdots - e\psi) = \cdots - \sum_{\rm sp}\int dV\ \int dW\ f\left(eJ_0\phi - \frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2\right)$$

  - vary with respect to ϕ (J_0 is Hermitian, do one $\nabla_\perp$ by parts, incorporate f into n):

$$\delta\left(\int dV\,\mathcal{L}\right) = -\sum_{\rm sp}\int dV\ \delta\phi\left(\int dW\ eJ_0 f + \nabla\cdot\frac{nmc^2}{B^2}\nabla_\perp\phi\right)$$

Polarisation (2): Ampere's Law

- find from Hamilton's principle, variation of $A_\|$

$$\int dV\,\mathcal{L} = \cdots + \sum_{\rm sp} \int dV \int dW\; f\left(\frac{e}{c}A_\| v_\|\right) - \int dV\,\frac{1}{8\pi}\left|\nabla_\perp A_\|\right|^2$$

  - vary with respect to $A_\|$ (do one $\nabla_\perp$ by parts):

$$\delta\left(\int dV\,\mathcal{L}\right) = \int dV\;\delta A_\| \left(\sum_{\rm sp}\left[\int dW\;\frac{e}{c}v_\|\,f\right] + \frac{1}{4\pi}\nabla_\perp^2 A_\|\right)$$

- resulting Ampere's law

$$\nabla_\perp^2 A_\| + \sum_{\rm sp}\frac{4\pi}{c}\int dW\; e v_\|\,f = 0$$

closed system: gyrokinetic equation for each f
and polarisation equations for both ϕ and $A_\|$

Energy Conservation

- main point:

  particle and field energy work together

- in this case, field energy controlling ϕ is actually ExB motion of particles
- an important auxiliary relation using polarisation:

$$\sum_{\rm sp} \int dV \int dW\; eJ_0\phi\, f = \sum_{\rm sp} \int dV\; nmv_E^2$$

- proof: multiply polarisation by ϕ, integrate over volume

$$\int dV\; \phi \sum_{\rm sp} \left[\int dW\; eJ_0 f + \nabla\cdot \frac{nmc^2}{B^2} \nabla_\perp \phi \right] = 0$$

$$\int dV\; \sum_{\rm sp} \left[\int dW\; \phi\, eJ_0 f - \nabla\phi\cdot \frac{nmc^2}{B^2} \nabla_\perp \phi \right] = 0$$

$$\int dV\; \sum_{\rm sp} \int dW\; f\, eJ_0\phi = \int dV\; \sum_{\rm sp} \frac{nmc^2}{B^2} \left|\nabla_\perp \phi\right|^2$$

ExB Energy

- “charge density” is really ExB energy
  - start with

$$\sum_{\mathrm{sp}} \int dV \int dW \; eJ_0\phi\, f = \sum_{\mathrm{sp}} \int dV \; nmv_E^2$$

  - subtract the ExB energy from both sides

$$\sum_{\mathrm{sp}} \int dV \left(\int dW \; eJ_0\phi\, f - nm\frac{v_E^2}{2} \right) = \sum_{\mathrm{sp}} \int dV \; nm\frac{v_E^2}{2} \equiv \mathcal{E}_E$$

  - combine left side into effective potential, move to right side

$$\mathcal{E}_E = \sum_{\mathrm{sp}} \int dV \int dW \; e\psi\, f$$

> ExB energy is equivalent to
> total gyrocenter potential energy

Magnetic Energy

- standard form in terms of magnetic field (integration of one $\nabla_\perp$ by parts)

$$\mathcal{E}_M = \int dV \; \frac{1}{8\pi} \left|\nabla_\perp A_\parallel\right|^2 = -\int dV \; \frac{1}{8\pi} \left(\nabla_\perp^2 A_\parallel\right) A_\parallel$$

- insert form from Ampere's law

$$\mathcal{E}_M = \sum_{\text{sp}} \frac{1}{2} \int dV \int dW \; e v_\parallel \, f \, \frac{1}{c} A_\parallel$$

- note that $v_\parallel = (1/m)\partial H/\partial z$, and do that derivative by parts (note this H includes $e\psi$)

$$\mathcal{E}_M = -\sum_{\text{sp}} \int dV \int dW \; H \, \frac{1}{2m} \frac{\partial f}{\partial v_\parallel} \frac{e}{c} A_\parallel$$

magnetic energy is equivalent to acceleration
by force potential in magnetic potential field

Thermal Energy

- this is simply the unperturbed Hamiltonian weighted by the particles

$$\mathcal{E}_F = \sum_{\mathrm{sp}} \int dV \int dW \; H_0 \, f$$

where

$$H_0 = \frac{m v_\parallel^2}{2} + \mu B$$

Total Energy

- Hamiltonian energy is therefore thermal plus ExB energy

$$\mathcal{E}_F + \mathcal{E}_E = \sum_{\text{sp}} \int dV \int dW \; H \, f = \sum_{\text{sp}} \int dV \int dW \; (H_0 + e\psi) \, f$$

- this adds to magnetic energy

$$\mathcal{E} = \mathcal{E}_F + \mathcal{E}_E + \mathcal{E}_M = \sum_{\text{sp}} \int dV \int dW \; H \left[f - \frac{1}{2m} \frac{\partial f}{\partial v_\parallel} \frac{e}{c} A_\parallel \right]$$

- part of energy due to H_0 is thermal energy
- part of energy due to $e\psi$ is ExB energy (“drift energy” or “electrostatic field energy”)
- part of energy due to $(e/c)A_\parallel$ is magnetic energy

Energy Conservation

- time derivatives of field energies work the same way (more manipulations):

$$\frac{\partial \mathcal{E}_E}{\partial t} = \frac{\partial}{\partial t} \sum_{\rm sp} \int dV \; nm \frac{v_E^2}{2} = \sum_{\rm sp} \int dV \int dW \; e\psi \, \frac{\partial f}{\partial t}$$

$$\frac{\partial \mathcal{E}_M}{\partial t} = \frac{\partial}{\partial t} \int dV \; \frac{1}{8\pi} \left| \nabla_\perp A_\parallel \right|^2 - \sum_{\rm sp} \int dV \int dW \; H \, \frac{e}{m} \frac{\partial f}{\partial v_\parallel} \, \frac{1}{c} \frac{\partial A_\parallel}{\partial t}$$

- so that the evolution of the total energy is given by . . .

$$\frac{\partial \mathcal{E}}{\partial t} = \sum_{\rm sp} \int dV \int dW \left[(H_0 + e\psi) \, \frac{\partial f}{\partial t} - H \, \frac{e}{m} \frac{\partial f}{\partial v_\parallel} \, \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \right]$$

$$= \sum_{\rm sp} \int dV \int dW \; H \left[\frac{\partial f}{\partial t} - \frac{e}{m} \frac{\partial f}{\partial v_\parallel} \, \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \right]$$

$$= \sum_{\rm sp} \int dV \int dW \; H[H, f] = 0$$

. . . zero!   One more little detail however . . .

Phase Space Conservation

... the assertion that the integral over $H[H,f]$ vanishes requires the Liouville theorem

- express the bracket in advection form

$$[H,f] = -V^\alpha \nabla_\alpha f \qquad \text{where} \qquad V^\alpha = \left\{ -\frac{c}{eBB_\parallel^*}\mathbf{F}\cdot\nabla H + \frac{\mathbf{b}^*}{m}\frac{\partial H}{\partial v_\parallel}, -\frac{1}{m}\mathbf{b}^*\cdot\nabla H \right\}$$

- it is important to note that not $\nabla_\alpha V^\alpha$ but $\nabla_\alpha(B_\parallel^* V^\alpha)$ is zero
  - ○ proof (use $\nabla\times\mathbf{b} = -\nabla\cdot(\mathbf{F}/B)$ and $\nabla\cdot\mathbf{B}^* = 0$):

$$\nabla_\alpha(B_\parallel^* V^\alpha) = -\nabla\cdot\left(\frac{c}{e}\frac{\mathbf{F}}{B}\cdot\nabla H\right) - \frac{1}{m}\frac{\partial\mathbf{B}^*}{\partial v_\parallel}\cdot\nabla H$$

$$= -\nabla\cdot\left(\frac{c}{e}\frac{\mathbf{F}}{B}\cdot\nabla H\right) - \frac{c}{e}(\nabla\times\mathbf{b})\cdot\nabla H$$

$$= -\nabla\cdot\left(\frac{c}{e}\frac{\mathbf{F}}{B}\cdot\nabla H\right) + \frac{c}{e}\left(\nabla\cdot\frac{\mathbf{F}}{B}\right)\cdot\nabla H$$

$$= 0$$

- the phase space element is $\int dW = 2\pi\int dv_\parallel\, d\mu\, B_\parallel^*$

Phase Space Conservation (2)

- thus we have

$$H[H,f] = \left[\frac{H^2}{2}, f\right] = -V_2^\alpha \nabla_\alpha f$$

  where V_2 is V defined with $H \to H^2/2$

- number and energy conservation follow the same way:

$$\int dV \int dW \; [H,f] = -2\pi \int dV \int dv_\parallel \int d\mu \, (B_\parallel^* V^\alpha) \nabla_\alpha f =$$

$$= -2\pi \int dV \int dv_\parallel \int d\mu \, \nabla_\alpha (f B_\parallel^* V^\alpha) = 0$$

  that is, zero.

- find energy conservation using $H \to H^2/2$ and $V \to V_2$ (trivial)

- find entropy conservation using $(\log f + 1)[H,f] = [H, f \log f]$

- note that the $(\partial f/\partial z)(\partial A_\parallel/\partial t)$ term conserves number and entropy trivially

Energy Transfer

- using total energy conservation it is simple to get the transfer, e.g., from thermal to ExB and magnetic energy, respectively:

$$\frac{\partial \mathcal{E}_F}{\partial t} = -\frac{\partial \mathcal{E}_E}{\partial t} - \frac{\partial \mathcal{E}_M}{\partial t}$$

$$\sum_{\mathrm{sp}} \int dV \int dW \; H_0 \frac{\partial f}{\partial t} = -\sum_{\mathrm{sp}} \int dV \int dW \left[e\psi \frac{\partial f}{\partial t} - e v_\parallel f \, \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \right]$$

- the energy theorem for zonal flows is found from the zonal contribution to ExB energy:
  - ◦ define the zonal average $\langle \phi \rangle$ according to coordinate system
  - ◦ field aligned, unit Jacobian Hamada coordinates x, y, s: $\quad \langle \phi \rangle = \int dy \int ds \, \phi$

$$\sum_{\mathrm{sp}} \int dx \int dW \; \langle e\psi \rangle \, \frac{\partial}{\partial t} \, \langle f \rangle = \text{“Reynolds stress”} + \text{“Maxwell stress”} - \text{“geodesic transfer”}$$

  - ◦ from the ExB and magnetic nonlinearities and curv/grad-B drifts, respectively

- gyrokinetic codes should be using relations like these as diagnostics!

The FEFI Model

- FEFI (Full Electrons Full Ions) code under construction: no delta-f approximations, Lagrangian is

$$L = \sum_{\rm sp} \int d\Lambda \, f \left[\frac{e}{c} \mathbf{A}^* \cdot \dot{\mathbf{R}} + \mu B \frac{\dot{\theta}}{\Omega} - H \right] - \int dV \, \frac{B_\perp^2}{8\pi}$$

where $d\Lambda = dW \, dV$ is the phase space element,
with v-space and volume elements dW and dV

- generalised magnetic potential

$$\mathbf{A}^* = \mathbf{A} + A_\parallel^* \mathbf{b} \qquad\qquad A_\parallel^* = A_\parallel + \frac{mc}{e} v_\parallel$$

with $\mathbf{B} = \nabla \times \mathbf{A} = B\mathbf{b}$ and $B_\parallel^* = \nabla \times \mathbf{A}^*$ and $B_\parallel^* = \mathbf{b} \cdot B_\parallel^*$

- generalised Hamiltonian

$$H = m \frac{v_\parallel^2}{2} + \mu B + e J_0 \phi - m \frac{v_E^2}{2}$$

- note v-space element includes Jacobian $B_\parallel^*$

$$\int d\Lambda = \int dV \int dW \qquad\qquad \int dW = \int dv_\parallel \, d\mu \, B_\parallel^*$$

- gyroaveraging

$$J_0 = \frac{\Pi_4(1 - x^2/a_n^2)}{\Pi_3(1 + b_n x^2)}$$

  approximates $J_0(x)$ to $x \approx 14$ with $x^2 = -\nabla \cdot \rho_L^2 \nabla_\perp$

- ExB velocity and perturbed magnetic field

$$v_E^2 = \frac{c^2}{B^2} \left|\nabla_\perp \phi\right|^2 \qquad B_\perp^2 = \left|\nabla_\perp A_\parallel\right|^2$$

- two important approximations
  - $A_\parallel$ is not gyroaveraged (belongs mostly to electrons)
  - low-$k_\perp$ form of screened potential (enters only at large scale)

FEFI Equations

- gyrokinetic equation

$$\frac{\partial f}{\partial t} - \frac{e}{mc}\frac{\partial f}{\partial v_\parallel}\frac{\partial A_\parallel}{\partial t} = [H, f] + C(f)$$

  with general Poisson bracket

$$B_\parallel^*[H, f] = \frac{\mathbf{B}^*}{m} \cdot \left(\frac{\partial f}{\partial v_\parallel}\nabla H - \frac{\partial H}{\partial v_\parallel}\nabla f\right) + \frac{c}{eB}\mathbf{F} : \nabla H \nabla f$$

  with magnetic stress tensor $\mathbf{F} = \nabla \mathbf{A} - (\nabla \mathbf{A})^T = \epsilon \cdot \mathbf{B}$
  and collision operator C added (see below)

- ExB polarisation, gyrokinetic Poisson equation

$$\sum_{\mathrm{sp}} \int dW \left[eJ_0 f + \nabla \cdot \frac{fmc^2}{B^2}\nabla_\perp \phi \right] = 0$$

- induction, gyrokinetic Ampere equation

$$\nabla_\perp^2 A_\parallel + \frac{4\pi}{c}\sum_{\mathrm{sp}} \int dW \, ev_\parallel \, f = 0$$

FEFI Energy

- thermal, ExB, and magnetic energy

$$\mathcal{E}_F = \sum_{\mathrm{sp}} \int d\Lambda \, H_0 \, f$$

$$\mathcal{E}_E = \sum_{\mathrm{sp}} \int d\Lambda \, f \, m \frac{v_E^2}{2} = \sum_{\mathrm{sp}} \int d\Lambda \, e\psi \, f$$

$$\mathcal{E}_M = \int dV \, \frac{B_\perp^2}{8\pi} = \sum_{\mathrm{sp}} \int d\Lambda \, \frac{e}{2c} A_\parallel v_\parallel \, f$$

where $H_0 = mv_\parallel^2/2 + \mu B$ and $e\psi = eJ_0\phi - mv_E^2/2$
using both field equations to evaluate $\mathcal{E}_E$ and $\mathcal{E}_M$

- time dependence of energy pieces

$$\frac{\partial \mathcal{E}_F}{\partial t} = \sum_{\mathrm{sp}} \int d\Lambda\, H_0 \frac{\partial f}{\partial t} \qquad\qquad \frac{\partial \mathcal{E}_E}{\partial t} = \sum_{\mathrm{sp}} \int d\Lambda\, e\psi \frac{\partial f}{\partial t}$$

and for $\mathcal{E}_M$ use $\partial H/\partial v_\parallel = m v_\parallel$

$$\frac{\partial \mathcal{E}_M}{\partial t} = \sum_{\mathrm{sp}} \int d\Lambda\, H \frac{e}{mc} \frac{\partial f}{\partial v_\parallel} \frac{\partial A_\parallel}{\partial t}$$

- energy conservation

$$\frac{\partial \mathcal{E}}{\partial t} = \sum_{\mathrm{sp}} \int d\Lambda\, H \left(\frac{\partial f}{\partial t} - \frac{e}{mc} \frac{\partial f}{\partial v_\parallel} \frac{\partial A_\parallel}{\partial t} \right)$$

$$= \sum_{\mathrm{sp}} \int d\Lambda\, H [H, f] = 0$$

requiring also that

$$\sum_{\mathrm{sp}} \int d\Lambda\, H\, C(f) = 0$$

FEFI Scheme for Induction

- how to handle $\partial A_\parallel/\partial t$, which must cancel $\nabla_\parallel\phi$ at large scale
- denote rest of equation as S

$$\frac{\partial f}{\partial t} - \frac{e}{mc}\frac{\partial f}{\partial v_\parallel}\frac{\partial A_\parallel}{\partial t} = S$$

- operate with integral in the induction equation

$$\left[(-\nabla_\perp^2) - \sum_{\rm sp}\int dW\,\frac{4\pi e^2}{mc^2}v_\parallel\frac{\partial f}{\partial v_\parallel}\right]\frac{\partial A_\parallel}{\partial t} = \frac{4\pi}{c}\sum_{\rm sp}\int dW\,ev_\parallel\,S$$

- solve this Helmholtz equation for time derivative, put on RHS

$$\frac{\partial f}{\partial t} = S + \frac{e}{mc}\frac{\partial f}{\partial v_\parallel}\frac{\partial A_\parallel}{\partial t}$$

- this RHS is actually an integro-differential of S

$$\frac{\partial f}{\partial t} = S + \frac{e}{mc}\frac{\partial f}{\partial v_\parallel}\mathcal{D}^{-1}\sum_{\rm sp}\int dW\,\frac{4\pi e}{c}v_\parallel\,S$$

where $\mathcal{D}$ is the operator in $[\cdots]$ above

$$\mathcal{D} = \left[(-\nabla_\perp^2) - \sum_{\rm sp}\int dW\,\frac{4\pi e^2}{mc^2}v_\parallel\frac{\partial f}{\partial v_\parallel}\right]$$

- evaluate for electrons only, at large scale ($\nabla_\perp^2 \to 0$)

$$\frac{\partial f}{\partial t} = S - \frac{e}{mc}\frac{\partial f}{\partial v_\parallel}\left(\int dW\,\frac{4\pi e^2}{mc^2}v_\parallel\frac{\partial f}{\partial v_\parallel}\right)^{-1}\int dW\,\frac{4\pi e}{c}v_\parallel\,S$$

- for $S = (e/m)(\nabla_\parallel\phi)(\partial f/\partial v_\parallel)$ the $v_\parallel$-moment of this cancels to machine accuracy

$$\int dW\,v_\parallel\frac{\partial f}{\partial t} = \int dW\,v_\parallel\,S - \int dW\,\frac{4\pi e^2}{mc^2}v_\parallel\frac{\partial f}{\partial v_\parallel}\left(\int dW\,\frac{4\pi e^2}{mc^2}v_\parallel\frac{\partial f}{\partial v_\parallel}\right)^{-1}\int dW\,v_\parallel\,S = 0$$

hence finite mass corrections are responsible for the dynamics

FEFI Geometry

- global version of flux tube form with shifted metric
- Hamada coordinates V, θ, ζ with the $B^\mu =$ functions of V only
- field aligning transform (minor radius a for normalisation)

$$x = V/a^3 \qquad y = q\theta - \zeta \qquad s = \theta$$

  with pitch parameter $q = B^\zeta / B^\theta$
  ○ set such that only B^s is nonzero

- coordinate range, finite chunk of torus, poloidal connection

$$x_0 < x < x_9 \qquad -\frac{1/2}{k_0} < y < \frac{1/2}{k_0} \qquad -\frac{1}{2} < s < \frac{1}{2}$$

  with radial boundaries x_0, x_9 and toroidal truncation integer k_0

- shifted metric transform

$$y_k = q(\theta - s_k) - \zeta - \Delta\alpha_k$$

  with s_k global constant and $\Delta\alpha_k$ a function of V only

- field aligning: only one B^μ is nonzero (B^s)

- perp dynamics on orthogonal grid: $g_k^{xy} = 0$ at each evaluation

- evaluate from simple model or Grad-Shafranov solution
  - ○ model profiles for pressure and current using HELENA code

- necessary quantities (functions of x and s)

$$g^{xx} \qquad g_k^{yy} \qquad \Delta\alpha_k \qquad B^s \qquad B$$

  giving also (i.a.) the components of magnetic stress tensor $\mathbf{F}$

- reference: B Scott, *Phys Plasmas* **8** (2001) 447

Collision Operator for FEFI

- like particles: pitch angle and $v_\parallel$ scattering

$$C_{jj} = \frac{\partial}{\partial v_\parallel} G_\parallel(v_\parallel) \left[(v_\parallel - U) + W \frac{\partial}{\partial v_\parallel} \right] + G(v) \frac{\partial}{\partial \zeta} \left(1 - \zeta^2 \right) \frac{\partial}{\partial \zeta}$$

  with U, W set to conserve energy, momentum
  - $G_\parallel$ and G are model potentials (including ν_j and v^{-3} dependence)
  - $\zeta = v_\parallel / v$ is the pitch angle
  - $v_\parallel$ and ζ are reckoned in c/m frame

- e-i: simple pitch angle scattering off the background
  - damps momentum to ions, conserves energy

- add a hypercollisionality ν_z to $C(f)$ for collisionless problems

$$C(f) = \cdots - \nu_z \frac{\partial^4 f}{\partial z^4}$$

  where $z = v_\parallel / V$ is the normalised coordinate (see below)

- normalised velocities used in FEFI

$$z = \frac{v_\parallel}{V} \qquad\qquad w = \frac{v_\perp^2}{2V^2} = \frac{\mu B}{V^2}$$

where V is a constant, noting v-space metric independent of B

- relation of variables (v, ζ) to (z, w)

$$z = v\zeta \qquad\qquad w = \frac{v^2}{2}\left(1 - \zeta^2\right)$$

where we also normalise v to V

- pitch angle derivative, note z commutes with $\partial/\partial w$

$$\frac{\partial}{\partial\zeta} = \frac{\partial z}{\partial\zeta}\frac{\partial}{\partial z} + \frac{\partial w}{\partial\zeta}\frac{\partial}{\partial w} = v\left(\frac{\partial}{\partial z} - z\frac{\partial}{\partial w}\right)$$

- do $G_\parallel$ and G pieces separately, also e-i for electrons
- set up in conservative finite volume form

Method for Conservation in Collisions

- evaluate v-space fluxes in cell centers, in 1-D in norm units

$$C_i = \frac{\partial}{\partial v_\|} G_\|(v_\|) \left[(v_\| - U) + W \frac{\partial}{\partial v_\|} \right] \Bigg|_{z=z_i} = \frac{1}{\Delta z} \left(\Gamma_{i-1} - \Gamma_i \right)$$

where $dz = z_i - z_{i-1}$

- momentum conservation

$$\sum_i z_i C_i = \sum_i \frac{z_i}{\Delta z} \left(\Gamma_{i-1} - \Gamma_i \right) = \sum_i \frac{z_{i+1} - z_i}{\Delta z} \Gamma_i = \sum_i \Gamma_i = 0$$

shifting index on Γ_{i-1} term

- energy conservation, uses z_i^2, note $z_{i+1}^2 - z_i^2 = \Delta z (z_{i+1} - z_i)$

$$\sum_i \frac{z_i^2}{2} C_i = \sum_i z_i \Gamma_i = 0$$

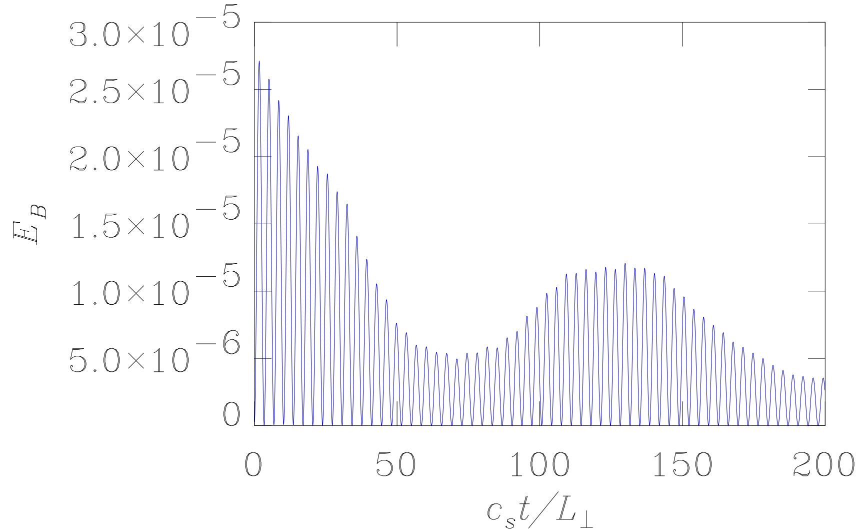
- two conditions for two unknowns U and W, coupled through $G_{\|}$
  - coefficients are various moments over combinations of f and $G_{\|}$

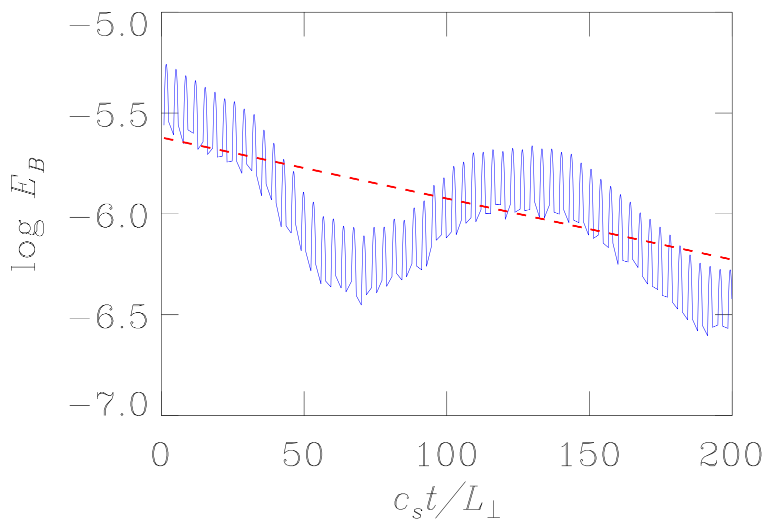
- pitch angle scattering set up to conserve momentum in 2-D
  - energy conservation follows automatically

- e-i pitch angle scattering damps current naturally
  - energy conservation follows if mean velocity for the background is zero

Kinetic Shear Alfvén Damping, Recursion

FEFI3, $\beta_e = 10^{-4}$ $\mu_e^{-1} = 3670$ $qR/L_\perp = 100$ $k_\perp \rho_s = 0.1$



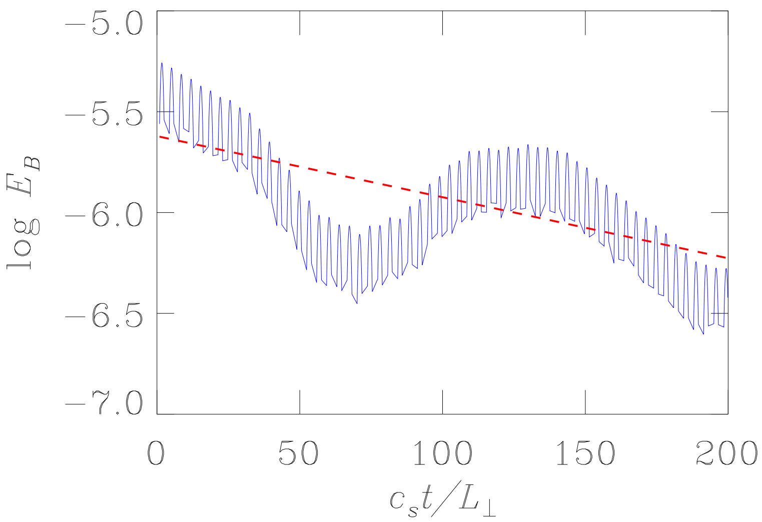


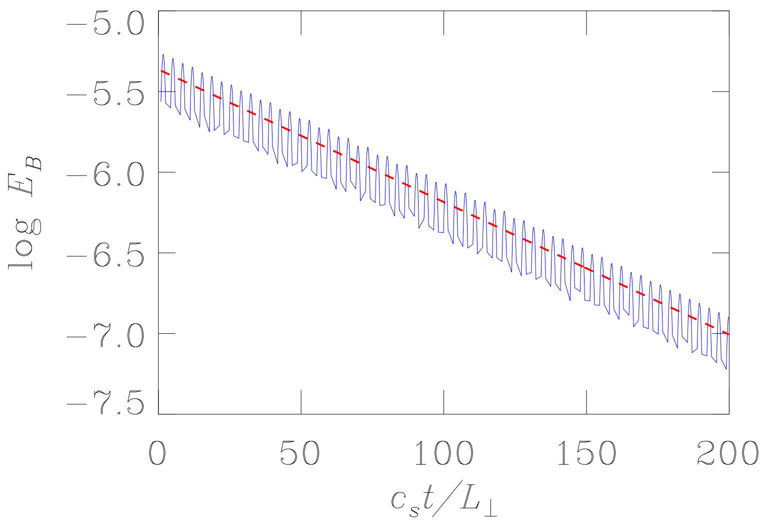
- recursion after $t = 40$, best seen in decay curve (right)
- non constant damping rate
- caused by velocity-space resolution problems (32 nodes in $v_\parallel$, with $\nu_z = 10^{-5}$)

Kinetic Shear Alfvén Damping, Physical

FEFI3, $\beta_e = 10^{-4}$ $\mu_e^{-1} = 3670$ $qR/L_\perp = 100$ $k_\perp \rho_s = 0.1$

- increase $v_\parallel$-resolution from 32 (left) to 64 (right) per $10\,V_e$



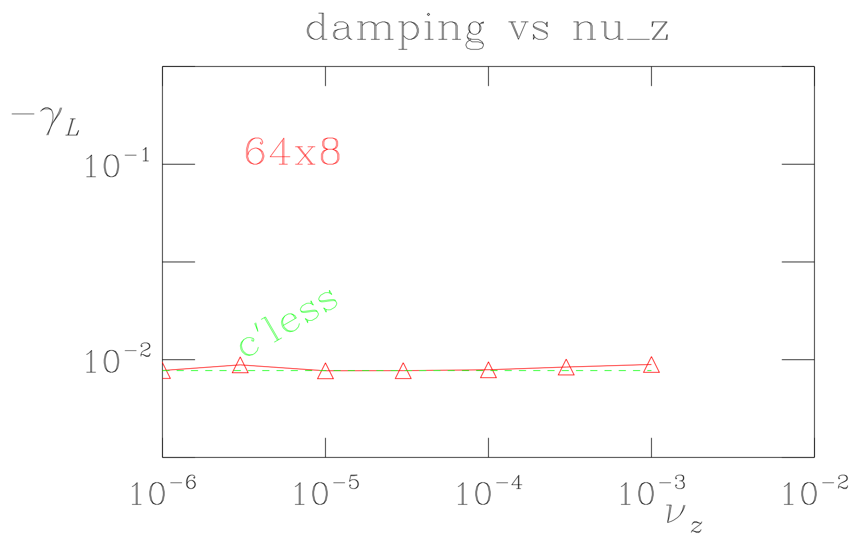


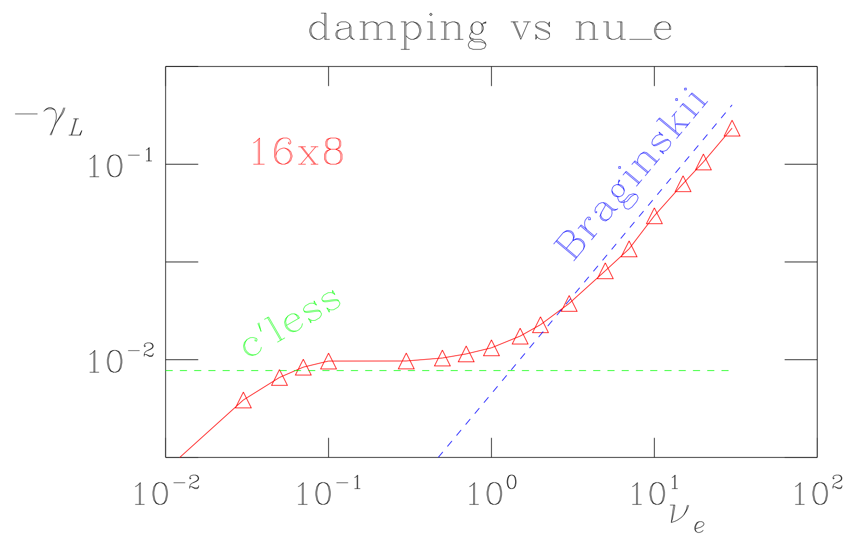
- well formed decay curve even at late times
- constant damping rate, good fit to part of curve near peaks
- velocity-space resolution sufficient (64 nodes in $v_\parallel$, with $\nu_z = 10^{-5}$)

KALF Damping Rate, collisionality scaling

FEFI3, $\beta_e = 10^{-4}$ $\mu_e^{-1} = 3670$ $qR/L_\perp = 100$ $k_\perp \rho_s = 0.1$

- collisionless (left) and trans-collisional (right)





- recursion problems for $\nu_z < 10^{-5}$
- non-thermalisation for $\nu_z > 10^{-3}$
- well converged for $N_z = 64$

- collisional regime for $\nu_e > 1$
- $v_\parallel$-space grid problems for $\nu_e < 0.1$
- usual range for edge turbulence is $\nu_e \sim 1$

Main Points

- **Gyrokinetic Equation**
  — Hamiltonian structure follows Lagrangian formulation
  — polarisation equation follows from field variation

- **Energy Conservation**
  — split between particle and field Hamiltonian
  — polarisation equation $\rightarrow$ ExB energy
  — magnetic potential $\rightarrow$ magnetic energy
  — simple bracket structure
  — depends on phase space conservation

- **FEFI Code**
  — large scale Alfvén problems solved
  — combination collisions/Alfvén/trapping conserves energy well
  — currently solving toroidal equilibrium problems, brackets, expect to finish in 2005

- **Main Targets**
  — nonlinear/nonlocal pedestal physics ($100\,\mathrm{eV} < T_e < 1\,\mathrm{keV}$)
  — edge and core zonal flow energetics
  — neoclassical tokamak equilibrium, perhaps also stellarators