Symmetry of Momentum Conservation in Gyrokinetics

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Outline

- **Gyrokinetic Field Theory**

  — system Lagrangian with canonical structure

  — symmetry in Lagrangian: time/energy, axisymmetry/momentum

- **Momentum Conservation in Gyrokinetic Theory**

  — canonical versus plasma momentum

  — phase space continuity, transport equations

  — role of charge conservation $\leftrightarrow$ FS-averaged radial current

- **Momentum Transport**

  — correspondence to the MHD limit

  — symmetric structure, dynamics (not spatial)

  — contributions of all orders have same symmetry constraints

Gyrokinetic Theory and Momentum

- since 2000:
  ◦ field theory (Sugama, Brizard 2000)
  ◦ major review: Brizard and Hahm, Rev Mod Phys (2007)
  ◦ dynamical "equilibrium" flows (Miyato and Scott 2009)
  ◦ full route to Maxwell equations (Pfirsch, Correa-Restrepo, Madsen 2005-10)

- momentum conservation and transport
  ◦ role of energetic consistency, vorticity conservation/transport (Scott 2010)
  ◦ role of dimensional reduction (GK) in momentum conservation (Brizard/Tromko 2011)

- any serious GK theory is a field theory with support by symmetry principles
  ◦ conservation from any Lagrangian in canonical form is rigorous
  ◦ proof of toroidal momentum transport involves vorticity transport equation

Basic Structure of GK Lagrangian

- symplectic part and Hamiltonian for particles, plus magnetic field energy

$$L = \sum_{\mathrm{sp}} \int d\Lambda\, f \left[\left(\frac{e}{c}\mathbf{A} + p_z \mathbf{b} - \frac{mc}{e}\mu \mathbf{W} \right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H \right] - \int d\mathcal{V}\, \frac{B_\perp^2}{8\pi}$$

- Hamiltonian depends on phase space coords and time/space-dependent fields

$$H = H(\mathbf{R}, p_z, \mu, \phi, A_\parallel) \qquad \{\phi, A_\parallel\} = \{\phi, A_\parallel\}(t, \mathbf{R}) \qquad \frac{\partial}{\partial\vartheta}(\text{anything}) = 0$$

- all field dependence is Lie-transformed into H (strictly)
  ◦ no $\partial/\partial t$ on fields in gyrokinetic (GK) eqn, no gyrophase (ϑ) dependence

- particle motion: Euler-Lagrange (E-L) eqs for gyrocenter coordinates

- GK eqn: Liouville theorem $\rightarrow$ distribution function f

- polarisation/induction eqs: E-L eqs for field potentials

Importance of Symmetry of GK Lagrangian

- symplectic part and Hamiltonian for particles, plus magnetic field energy

$$L = \sum_{\rm sp} \int d\Lambda\, f \left[\left(\frac{e}{c}\mathbf{A} + p_z \mathbf{b} - \frac{mc}{e}\mu \mathbf{W} \right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H \right] - \int d\mathcal{V}\, \frac{B_\perp^2}{8\pi}$$

- all dependent variable involvement is in time component only
- symplectic part is strictly static and axisymmetric (contains geometry information)
- conservation laws follow: time symmetry $\rightarrow$ energy conservation
- toroidal angle (axi-)symmetry $\rightarrow$ toroidal momentum conservation
- all approximations appear in the derivation of the Lagrangian (Lie transforms)
- approximations are subect to exact preservation of the symmetry
- if this symmetry setup is rigorous, the results are rigorous

Lagrangian Density

- we can define a Lagrangian density $\mathcal{L}$ as follows

$$L \equiv \int d\mathcal{V}\,\mathcal{L}$$

- hence with the particles we have the species sum and velocity integration

$$\mathcal{L} = \mathcal{L}_f + \sum_{\mathrm{sp}} \int d\mathcal{W}\, f\, L_p$$

- here, $\mathcal{L}_f$ is the *field Lagrangian density* and L_p is the *particle Lagrangian*

$$L_p = \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b} - \frac{mc}{e}\mu\mathbf{W}\right)\cdot\dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H$$

$$\mathcal{L}_f = -\frac{B_\perp^2}{8\pi}$$

- electric energy enters H only (ExB kinetic), not $\mathcal{L}_f$ (electric field)
  - this is *quasineutrality*

Canonical Toroidal Momentum Distribution

- distribution function f satisfies Liouville Theorem

$$\dot{f} = 0 \implies \frac{\partial f}{\partial t} + \dot{\mathbf{Z}}_p \cdot \frac{\partial f}{\partial \mathbf{Z}_p} = 0$$

- incompressible form with phase space volume element

$$\frac{\partial}{\partial t}(f\sqrt{g}B_\parallel^*) + \frac{\partial}{\partial \mathbf{Z}_p} \cdot \left(f\sqrt{g}B_\parallel^* \dot{\mathbf{Z}}_p\right) = 0$$

- using this, multiply equation for P_φ by $fB_\parallel^*$ to find (using $\dot{P}_\varphi = -\partial H/\partial\varphi$)

$$\frac{\partial}{\partial t}(f\sqrt{g}B_\parallel^* P_\varphi) + \frac{\partial}{\partial \mathbf{Z}_p} \cdot \left(f\sqrt{g}B_\parallel^* P_\varphi \dot{\mathbf{Z}}_p\right) = -(f\sqrt{g}B_\parallel^*)\frac{\partial H}{\partial \varphi}$$

- this is the simple derivation of toroidal momentum phase space continuity

Result using Functional Derivatives

- assume $H = H(A, \nabla A)$ and find (species sum and velocity integration understood)

$$f\frac{\partial H}{\partial \varphi} = f\frac{\partial H}{\partial A}\frac{\partial A}{\partial \varphi} + f\frac{\partial H}{\partial \nabla A}\cdot\nabla\frac{\partial A}{\partial \varphi}$$

$$= \left(f\frac{\partial H}{\partial A} - \nabla\cdot f\frac{\partial H}{\partial \nabla A}\right)\frac{\partial A}{\partial \varphi} + \nabla\cdot f\frac{\partial H}{\partial \nabla A}\frac{\partial A}{\partial \varphi}$$

- use field equation $\delta L/\delta A = 0$ to replace with Lagrangian densities $\mathcal{L}$ and $\mathcal{L}_f$

$$f\frac{\partial H}{\partial \varphi} = \left(\frac{\partial \mathcal{L}_f}{\partial A} - \nabla\cdot\frac{\partial \mathcal{L}_f}{\partial \nabla A}\right)\frac{\partial A}{\partial \varphi} + \nabla\cdot\frac{\partial \mathcal{L}_f}{\partial \nabla A}\frac{\partial A}{\partial \varphi} - \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla A}\frac{\partial A}{\partial \varphi}$$

- combine the $\mathcal{L}_f$ pieces

$$f\frac{\partial H}{\partial \varphi} = \left(\frac{\partial \mathcal{L}_f}{\partial A} + \frac{\partial \mathcal{L}_f}{\partial \nabla A}\cdot\nabla\right)\frac{\partial A}{\partial \varphi} - \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla A}\frac{\partial A}{\partial \varphi}$$

- finally

$$f\frac{\partial H}{\partial \varphi} = \frac{\partial \mathcal{L}_f}{\partial \varphi} - \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla A}\frac{\partial A}{\partial \varphi}$$

Canonical Toroidal Momentum Conservation

- simplified version assumes field dependence like $H(\phi, A_\parallel, \nabla\phi, \nabla A_\parallel)$
  - (general: add $\nabla_\perp^2$ dependence, done in paper)

- apply velocity space integral to annihilate $\partial/\partial p_z$ and $\partial/\partial\mu$
  - note that $\int d\mathcal{W}/B_\parallel^*$ commutes past ∇ and that $(\partial/\partial\mathbf{Z}_p)\cdot(\sqrt{g}\mathbf{S}) = \sqrt{g}\nabla\cdot\mathbf{S}$

$$\frac{\partial}{\partial t}\sum_{\text{sp}}\int d\mathcal{W}\, f P_\varphi + \nabla\cdot\sum_{\text{sp}}\int d\mathcal{W}\, f P_\varphi \dot{\mathbf{R}} = -\sum_{\text{sp}}\int d\mathcal{W}\, f\frac{\partial H}{\partial\varphi}$$

- use the result on prev page to evaluate $\partial H/\partial\varphi$ term

$$\sum_{\text{sp}}\int d\mathcal{W}\, f\frac{\partial H}{\partial\varphi} = \frac{\partial\mathcal{L}_f}{\partial\varphi} - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi}\frac{\partial\phi}{\partial\varphi} + \frac{\partial\mathcal{L}}{\partial\nabla A_\parallel}\frac{\partial A_\parallel}{\partial\varphi}\right)$$

- integration over space annihilates $\partial/\partial\varphi$ and $\nabla\cdot$, yielding **momentum conservation**

$$\frac{\partial}{\partial t}\sum_{\text{sp}}\int d\Lambda\, f P_\varphi = 0$$

Energy Conservation

- simplified version assumes field dependence like $H(\phi, A_\|, \nabla\phi, \nabla A_\|)$
  - (general: add $\nabla_\perp^2$ dependence, done in paper)

- same calculation as before, but with H and $\partial/\partial t$ instead of P_φ and $\partial/\partial\varphi$
  - some bracket manipulation to get $fH\dot{\mathbf{R}}$ together under the divergence

$$\frac{\partial}{\partial t}\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH + \nabla\cdot\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH\dot{\mathbf{R}} = \sum_{\mathrm{sp}}\int d\mathcal{W}\, f\frac{\partial H}{\partial t}$$

- evaluate the last term the same way as before
  - note the $\partial\mathcal{L}_f/\partial t$ term survives (note sign!)

- this yields **energy conservation**
  - dependence on $A_\|$ gives magnetic energy, on ϕ gives ExB energy

$$\frac{\partial}{\partial t}\left(\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH - \mathcal{L}_f\right) + \nabla\cdot\left(\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH\dot{\mathbf{R}} + \frac{\partial\mathcal{L}}{\partial\nabla\phi}\frac{\partial\phi}{\partial t} + \frac{\partial\mathcal{L}}{\partial\nabla A_\|}\frac{\partial A_\|}{\partial t}\right) = 0$$

Canonical Toroidal Momentum Transport

- simplified version assumes field dependence like $H(\phi, A_\parallel, \nabla\phi, \nabla A_\parallel)$
  - (general: add $\nabla_\perp^2$ dependence, done in paper)
- use the functional derivative result to evaluate the $\partial f H/\partial\varphi$ term

$$\sum_{\rm sp} \int d\mathcal{W}\, f \frac{\partial H}{\partial \varphi} = \frac{\partial \mathcal{L}_f}{\partial \varphi} - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \nabla\phi}\frac{\partial \phi}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial \nabla A_\parallel}\frac{\partial A_\parallel}{\partial \varphi}\right)$$

- the flux surface average will annihilate $\partial/\partial\varphi$
- this will leave only the divergence term
- the $\partial f H/\partial\varphi$ term gives ExB Reynolds and Maxwell stresses generally
  - which add to the direct perp/parallel transport terms
- first we will define the flux surface average ...

Flux Surface Average

- average over all phase space coordinates except volume V
  - species sum is included (and in below forms, understood)

- flux surface average annihilates divergences except $\partial/\partial V$, commutes past $\partial/\partial t$
  - $\int d\mathcal{W}/B_\parallel^*$ commutes past all spatial derivatives, then do the angle derivatives

- flux surface average of kinetic quantities implies the species sum, hence …

$$\frac{\partial}{\partial t}(f\sqrt{g}B_\parallel^* P_\varphi) + \frac{\partial}{\partial \mathbf{Z}_p} \cdot \left(f\sqrt{g}B_\parallel^* P_\varphi \dot{\mathbf{Z}}_p\right) = -(f\sqrt{g}B_\parallel^*)\frac{\partial H}{\partial \varphi}$$

… becomes …

$$\frac{\partial}{\partial t}\left\langle f P_\varphi \right\rangle + \frac{\partial}{\partial V}\left\langle f P_\varphi \dot{V} \right\rangle + \left\langle f \frac{\partial H}{\partial \varphi} \right\rangle = 0$$

- evaluate the last term, include in the divergence

$$\frac{\partial}{\partial t}\left\langle f P_\varphi \right\rangle + \frac{\partial}{\partial V}\left(\left\langle f P_\varphi \dot{V} \right\rangle - \left\langle \frac{\partial \mathcal{L}}{\partial \nabla\phi}\frac{\partial \phi}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial \nabla A_\parallel}\frac{\partial A_\parallel}{\partial \varphi} \right\rangle\right) = 0$$

Plasma Momentum — Polarisation

- canonical momentum under quasineutrality is just the plasma momentum
- define the polarisation and the generalised vorticity . . .

$$\mathbf{P} = -\frac{\partial \mathcal{L}}{\partial \nabla \phi} \qquad\qquad \Omega \equiv -\nabla \cdot \mathbf{P}$$

. . . write the charge conservation equation . . .

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{P} + \nabla \cdot \left(\sum_{\mathrm{sp}} \int d\mathcal{W}\, f e \dot{\mathbf{R}} \right) = 0$$

. . . then take the flux surface average to get the vorticity transport equation

$$\frac{\partial}{\partial V} \left\langle \frac{\partial P^V}{\partial t} + f e \dot{V} \right\rangle = 0$$

Plasma Momentum — Radial Current

- the vorticity transport equation is

$$\frac{\partial}{\partial V}\left\langle \frac{\partial P^V}{\partial t} + fe\dot{V} \right\rangle = 0$$

- assume to vanish at $V = 0$, take the $\partial/\partial V$ off

$$\left\langle \frac{\partial P^V}{\partial t} + fe\dot{V} \right\rangle = 0$$

- this is the statement that the flux surface averaged radial current vanishes

we will use this to convert canonical momentum
to plasma momentum in the transport

Plasma Momentum — ExB Component

- content of canonical momentum is the phase space integral, split into pieces

$$\mathcal{P} = \sum_{\text{sp}} \int d\Lambda\, f\, P_\varphi = \sum_{\text{sp}} \int d\Lambda\, f \left(\frac{e}{c} A_\varphi + p_z b_\varphi \right)$$

pull the A_φ flux fuction out and take the flux surface average

$$\mathcal{P} = \int dV \left(\frac{1}{c} A_\varphi \left\langle fe \right\rangle + \left\langle f p_z b_\varphi \right\rangle \right)$$

use $\nabla \cdot \mathbf{P}$ for fe and evaluate

$$\mathcal{P} = \int dV \left(-\frac{1}{c} \left\langle \mathbf{P} \cdot \nabla A_\varphi \right\rangle + \left\langle f p_z b_\varphi \right\rangle \right)$$

- in an MHD model you can show directly the $\mathbf{P} \cdot \nabla A_\varphi$ piece is ExB momentum

$$\rho_M \frac{c}{B^2} \nabla\phi \cdot \nabla A_\varphi = \cdots = \rho_M (\mathbf{v}_E)_\varphi$$

(detail on the ExB momentum)

- use magnetic field representation

$$\mathbf{B} = I\nabla\varphi + \nabla A_\varphi \times \nabla\varphi$$

- evaluate

$$\frac{c}{B^2}\nabla\phi \cdot \nabla A_\varphi = \frac{c}{B^2}R^2(\nabla\phi\times\nabla\varphi)\cdot(\nabla A_\varphi\times\nabla\varphi)$$

$$= \frac{c}{B^2}R^2(\nabla\phi\times\nabla\varphi)\cdot\mathbf{B}$$

$$= \frac{c}{B^2}R^2(\mathbf{B}\times\nabla\phi)\cdot\nabla\varphi$$

$$\frac{c}{B^2}\nabla\phi \cdot \nabla A_\varphi = R^2\mathbf{v}_E\cdot\nabla\varphi$$

- finally lower the toroidal angle index

$$\frac{c}{B^2}\nabla\phi \cdot \nabla A_\varphi = (\mathbf{v}_E)_\varphi$$

Canonical Momentum — Polarisation Cancellation

- continuity equation for canonical momentum

$$\frac{\partial}{\partial t}\sum_{\rm sp}\int d\mathcal{W}\,fP_\varphi + \nabla\cdot\sum_{\rm sp}\int d\mathcal{W}\,fP_\varphi\dot{\mathbf{R}} = -\sum_{\rm sp}\int d\mathcal{W}\,f\frac{\partial H}{\partial\varphi}$$

take flux surface average (remember species sum)

$$\frac{\partial}{\partial t}\left\langle fP_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle fP_\varphi\dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle$$

look at the terms involving A_φ (which pulls out of the FS average)

$$\frac{\partial}{\partial t}\frac{A_\varphi}{c}\left\langle fe\right\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle = \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle\frac{\partial P^V}{\partial t} + fe\dot{V}\right\rangle - \frac{\partial}{\partial t}\frac{1}{c}\left\langle\mathbf{P}\cdot\nabla A_\varphi\right\rangle$$

the first term on the RHS vanishes (radial current)
leaving just the plasma momentum

Toroidal Momentum Transport Equation

- we now have the terms with the plasma momentum and transport

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi \right\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V} \right\rangle = -\left\langle f \frac{\partial H}{\partial \varphi} \right\rangle$$

- put in the above evaluations for the right side

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi \right\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V} - \frac{\partial \mathcal{L}}{\partial \nabla\phi}\frac{\partial \phi}{\partial \varphi} - \frac{\partial \mathcal{L}}{\partial \nabla A_\parallel}\frac{\partial A_\parallel}{\partial \varphi} \right\rangle = 0$$

- the transport terms are ExB and magnetic flutter transport of parallel momentum
  - plus the field terms which give ExB Renolds and Maxwell stresses

there are no spurious terms due to canonical pieces

Summary of Momentum Conservation/Transport Results

- the theory has a rigorous basis in Hamilton's Principle and symmetry
- time/toroidal-angle symmetry give energy/toroidal-momentum conservation
- polarisation cancellation yields plasma momentum from canonical momentum
  - Poynting momentum gets dropped by quasineutrality assumptions (small $E^2/8\pi$)
- a well formed toroidal plasma momentum transport equation results
  - no spurious terms due to A_φ (canonical forms)
- hence (in paper) no considerations of higher order pieces
- main reference:

B Scott and J Smirnov, Phys Plasmas 17 (2010) 112302
Energetic Consistency and Momentum Conservation in the Gyrokinetic Description of Tokamak Plasmas

Reynolds Stress and Momentum Transport

- the toroidal momentum transport equation is

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V} - \frac{\partial \mathcal{L}}{\partial \nabla\phi}\frac{\partial\phi}{\partial\varphi} - \frac{\partial \mathcal{L}}{\partial \nabla A_\parallel}\frac{\partial A_\parallel}{\partial\varphi}\right\rangle = 0$$

- using an "MHD Lagrangian" in a mean-field model (paper) this is evaluated as

$$\frac{\partial}{\partial t}\left\langle \rho_M u_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle\left(\rho_M \widetilde{v}_E^V \widetilde{u}_\parallel + \widetilde{p}\,\widetilde{b}^V\right) b_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle \rho_M \widetilde{v}_E^V \widetilde{v}_{E\varphi}\right\rangle + \frac{\partial}{\partial V}\left\langle \rho_M v_A^2 \widetilde{b}^V \widetilde{b}_\varphi\right\rangle = 0$$

- Reynolds stress(RS): same sign as in vorticity transport equation
  - ◦ if RS drives the flow, this represents negative diffusivity
  - ◦ RS only transports momentum if it takes energy out of the flow

- hence one expects the parallel/radial component to be most relevant
  - ◦ this component functions if turbulence spatial symmetry is broken
  - ◦ otherwise (even ϕ odd $u_\parallel$ poloidally) the necessary correlation vanishes

A Result on Symmetry

- Reynolds stress has a symmetric structure, easiest to see in fluxtube geometry:

$$\text{for} \quad H = H(\phi, |\nabla\phi|^2) \quad \text{define} \quad \frac{\partial \mathcal{L}}{\partial \nabla\phi} \equiv N\nabla\phi$$

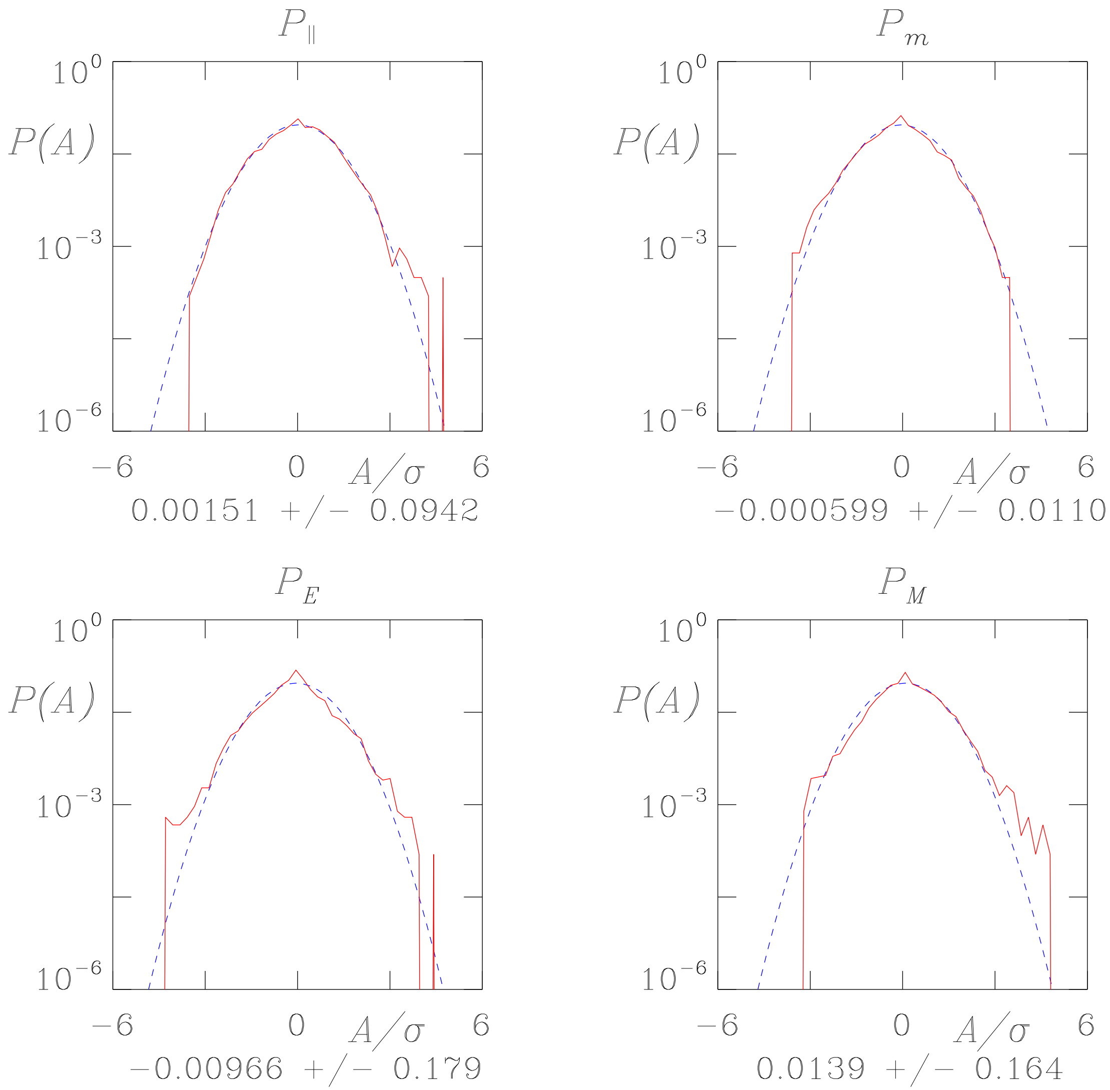
$$\nabla \cdot \left\langle -\frac{\partial \mathcal{L}}{\partial \nabla\phi} \frac{\partial \phi}{\partial \varphi} \right\rangle = \nabla \cdot \left\langle -N\nabla\phi \frac{\partial \phi}{\partial \varphi} \right\rangle \qquad \text{becomes} \qquad -\frac{\partial}{\partial x} \left\langle N \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \right\rangle$$

$$\text{expression of symmetry:} \quad -\frac{1}{2} \left\langle N \left(\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial x} \right) \right\rangle$$

- plus sign, exchange of derivatives $\partial/\partial x$ and $\partial/\partial y$
- vanishes for spatially symmetric eigenmodes, but basic result is different:
- **dynamical symmetry:** nonzero, but with symmetric PDF with zero mean
- demonstration: delta-FEFI edge case
  (you only need a fluid model but this is what I have to hand)

FS avg momentum flux distributions

delta-FEFI, L-mode Base Case, $\hat{\beta} = 2.39$, $\hat{\mu} = 10.0$, $C = 3.60$



Main Point – Symmetry

- these are estimates from the gyrofluid moments found from δf

$$P_{\|} = \langle n_i M_i \widetilde{u}_{\|} \widetilde{v}_E^x \rangle \qquad P_m = \left\langle \widetilde{p}_{i\|} \widetilde{b}^x \right\rangle$$

$$P_E = \langle n_i M_i \widetilde{v}_E^x \widetilde{v}_E^y \rangle \qquad P_M = \left\langle n_i M_i v_A^2 \widetilde{b}^x \widetilde{b}^y \right\rangle$$

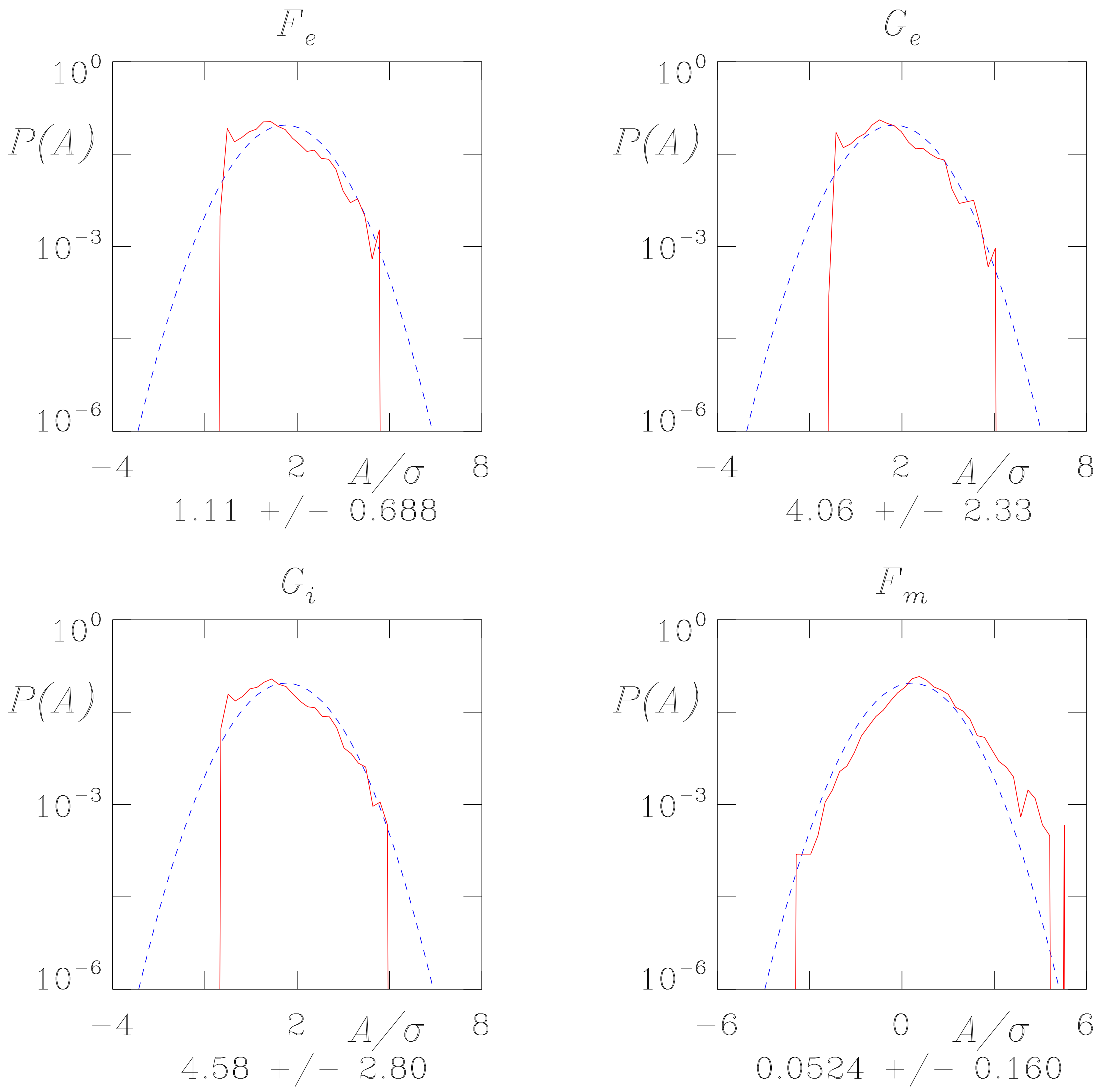
- PDFs are very close to zero mean
- zonal Reynolds stress does not vanish but does average to small values
- the RMS levels (the std dev) are however much smaller than the thermal fluxes

$$F_e = \langle \widetilde{n}_e \widetilde{v}_E^x \rangle \qquad G_e = \frac{3}{2} n_e \left\langle \widetilde{T}_e \widetilde{v}_E^x \right\rangle \qquad G_i = \frac{3}{2} n_i \left\langle \widetilde{T}_i \widetilde{v}_E^x \right\rangle \qquad F_m = \left\langle \widetilde{q}_{e\|} \widetilde{b}^x \right\rangle$$

- the thermal fluxes have strong average values (mean > std dev)

FS avg thermal flux distributions

delta-FEFI, L-mode Base Case, $\hat{\beta} = 2.39$, $\hat{\mu} = 10.0$, $C = 3.60$



The Laplacian Term and Reynolds Stress

- allow $H = H(\phi, |\nabla\phi|^2, \nabla_\perp^2\phi) = e\phi + \widehat{H}(|\nabla\phi|^2, \nabla_\perp^2\phi)$, no background terms, then

$$\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle = \left\langle fe\frac{\partial\phi}{\partial\varphi} + f\frac{\partial H}{\partial\nabla_\perp\phi}\nabla_\perp\frac{\partial\phi}{\partial\varphi} + f\frac{\partial H}{\partial\nabla_\perp^2\phi}\nabla_\perp^2\frac{\partial\phi}{\partial\varphi}\right\rangle$$

integrate by parts to set up the functional derivative

$$\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle = \left\langle\left(fe - \nabla\cdot f\frac{\partial H}{\partial\nabla_\perp\phi} + \nabla_\perp^2 f\frac{\partial H}{\partial\nabla_\perp^2\phi}\right)\frac{\partial\phi}{\partial\varphi}\right\rangle$$

$$+\left\langle\nabla\cdot\left(f\frac{\partial H}{\partial\nabla_\perp\phi}\frac{\partial\phi}{\partial\varphi} + f\frac{\partial H}{\partial\nabla_\perp^2\phi}\nabla_\perp\frac{\partial\phi}{\partial\varphi} - \frac{\partial\phi}{\partial\varphi}\nabla_\perp f\frac{\partial H}{\partial\nabla_\perp^2\phi}\right)\right\rangle$$

first line vanishes (no background terms in L_f), leaving

$$\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle = \left\langle\nabla\cdot\left(f\frac{\partial H}{\partial\nabla_\perp\phi}\frac{\partial\phi}{\partial\varphi} + f\frac{\partial H}{\partial\nabla_\perp^2\phi}\nabla_\perp\frac{\partial\phi}{\partial\varphi} - \frac{\partial\phi}{\partial\varphi}\nabla_\perp f\frac{\partial H}{\partial\nabla_\perp^2\phi}\right)\right\rangle$$

this was the result from the 2010 paper:

$$\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle = \left\langle \nabla\cdot\left(f\frac{\partial H}{\partial\nabla_\perp\phi}\frac{\partial\phi}{\partial\varphi} + f\frac{\partial H}{\partial\nabla_\perp^2\phi}\nabla_\perp\frac{\partial\phi}{\partial\varphi} - \frac{\partial\phi}{\partial\varphi}\nabla_\perp f\frac{\partial H}{\partial\nabla_\perp^2\phi}\right)\right\rangle$$

now we do the $\partial/\partial\varphi$ by parts in the second term inside $\langle\,\rangle$

$$\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle = \left\langle \nabla\cdot\left(f\frac{\partial H}{\partial\nabla_\perp\phi}\frac{\partial\phi}{\partial\varphi} - \frac{\partial}{\partial\varphi}\left[f\frac{\partial H}{\partial\nabla_\perp^2\phi}\right]\nabla_\perp\phi - \frac{\partial\phi}{\partial\varphi}\nabla_\perp\left[f\frac{\partial H}{\partial\nabla_\perp^2\phi}\right]\right)\right\rangle$$

hence the FLR term (the $\nabla_\perp^2\phi$ dependence) exhibits full symmetry

- diamagnetic Reynolds stress in symmetric form

$$P_E \equiv \sum_{\mathrm{sp}}\int d\mathcal{W}\, f\frac{\partial H}{\partial\nabla_\perp^2\phi} \qquad \text{then} \qquad \Pi^V_{FLR} = -\left\langle \nabla V\cdot\left(\frac{\partial P_E}{\partial\varphi}\nabla_\perp\phi + \frac{\partial\phi}{\partial\varphi}\nabla_\perp P_E\right)\right\rangle$$

in a fluxtube model with unit diagonal metric ("shifted metric" PoP 6/1998)

$$\Pi^x_{FLR} = -\left\langle \frac{\partial P_E}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial P_E}{\partial x}\right\rangle$$

same thing as diamagnetic Reynolds stress in a reduced Braginskii model

Symmetry with FLR Effects

- main addition: FLR correction to H, polarisation, diamagnetic Reynolds stress

$$H = e\phi + \frac{\rho_L^2}{4}\nabla_\perp^2(e\phi) - \frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2$$

$$\sum_{\rm sp}\left[en + \nabla_\perp^2\frac{mc^2p_\perp}{2eB^2} + \nabla\cdot\frac{nmc^2}{B^2}\nabla_\perp\phi\right] = 0$$

$$P_E \equiv \sum_{\rm sp}\frac{mc^2p_\perp}{2eB^2} \qquad \text{then} \qquad \Pi^x_{FLR} = -\left\langle\frac{\partial P_E}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial P_E}{\partial x}\right\rangle$$

Symmetry with FLR and ExB-Mach Effects

- further addition: ExB correction to FLR term (Miyato et al JPSJ 2009)

$$H = e\phi + \frac{\rho_L^2 + \rho_E^2}{4} \nabla_\perp^2 (e\phi) - \frac{mc^2}{2B^2} \left|\nabla_\perp \phi\right|^2$$

$$\rho_{GY} = \sum_{\mathrm{sp}} ne \qquad\qquad \left[\rho_{GY} + \nabla_\perp^2 P_E + \nabla \cdot N \nabla_\perp \phi\right] = 0$$

now with ExB Mach (u_E^2) and vorticity (Ω_E) corrections

$$N \equiv \sum_{\mathrm{sp}} \frac{nmc^2}{B^2} \left(1 - \frac{\Omega_E}{2\Omega}\right) \qquad\qquad P_E \equiv \sum_{\mathrm{sp}} \frac{mc^2}{2eB^2} \left(p_\perp + \frac{1}{2} nmu_E^2\right)$$

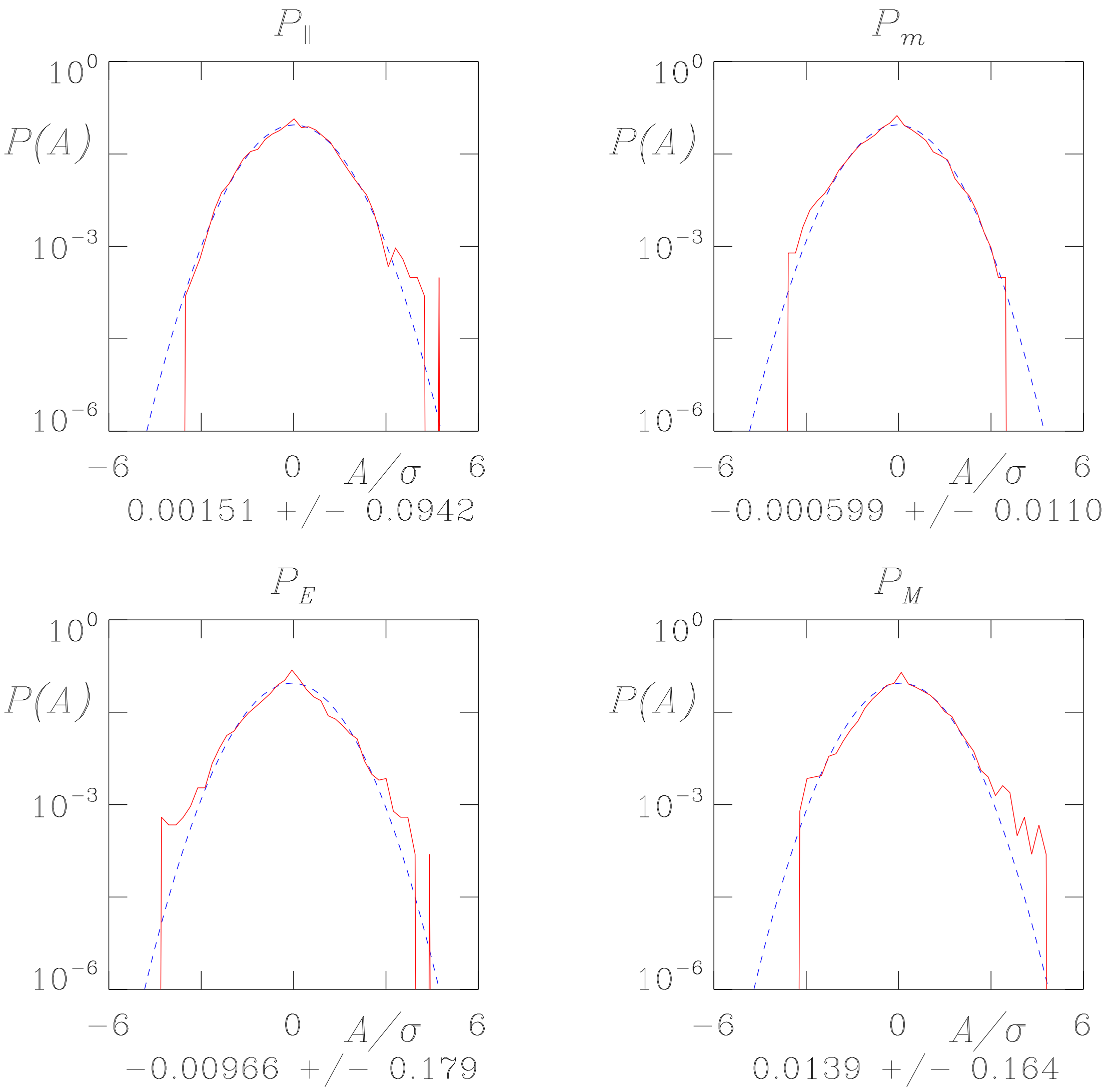
both pieces of Reynolds stress

$$\Pi^x = -\left\langle N \left(\frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial x}\right)\right\rangle - \left\langle \frac{\partial P_E}{\partial y} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial P_E}{\partial x}\right\rangle$$

- in this form main symmetry breaker is N
  - mainly: density gradient and flow equilibrium
  - secondary: ExB vorticity correction to N

FS avg momentum flux distributions

delta-FEFI, L-mode Base Case, $\hat{\beta} = 2.39$, $\hat{\mu} = 10.0$, $C = 3.60$



Higher-order Pieces

- examine the higher-order corrections
- inhomogeneity correction and diamagnetic Reynolds stress

$$P_{0E} = n_0'' \frac{x^2}{2} \left\langle n_i M_i \widetilde{v}_E^x \widetilde{v}_E^y \right\rangle \qquad P_{1E} = \frac{1}{2} \left\langle n_i M_i (\widetilde{u}_*^x \widetilde{v}_E^y + \widetilde{v}_E^x \widetilde{u}_*^y) \right\rangle$$

with diamagnetic ion velocity

$$\widetilde{\mathbf{u}}_* = -\frac{1}{n_i e} \widehat{\mathbf{F}} \cdot \nabla \widetilde{p}_i$$

- pieces from vorticity and ExB Mach corrections

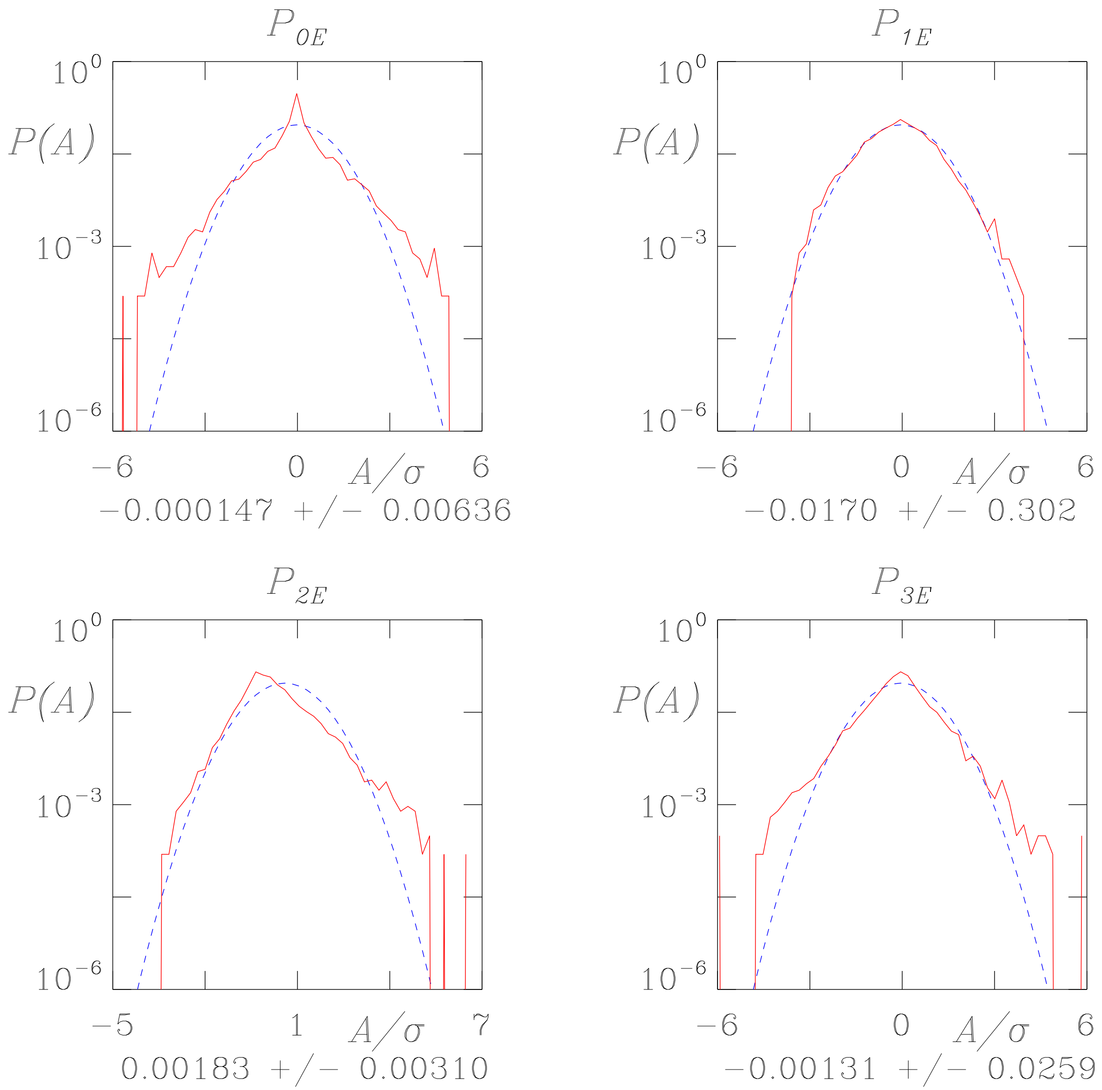
$$P_{2E} = \left\langle n_i M_i \frac{\widetilde{n}_e - \widetilde{n}_i}{2 n_e} \widetilde{v}_E^x \widetilde{v}_E^y \right\rangle \qquad P_{3E} = \frac{1}{2} \left\langle n_i M_i (\widetilde{w}_E^x \widetilde{v}_E^y + \widetilde{v}_E^x \widetilde{w}_E^y) \right\rangle$$

with ExB Mach correction to the diamagnetic ion velocity

$$\widetilde{\mathbf{w}}_E = -\frac{M_i}{e} \widehat{\mathbf{F}} \cdot \nabla \frac{u_E^2}{2}$$

FS avg higher-order momentum flux distributions

delta-FEFI, L-mode Base Case, $\hat{\beta} = 2.39$, $\hat{\mu} = 10.0$, $C = 3.60$



Symmetrisation of Main Reynolds Stress

- start with

$$\Pi_E^x = -\left\langle N \left(\frac{\partial\phi}{\partial y} \frac{\partial\phi}{\partial x} \right) \right\rangle$$

re-arrange $\partial/\partial x$ and $\partial/\partial y$ alternatively

$$\Pi_E^x = -\frac{1}{2} \left\langle N \left(\frac{\partial}{\partial y} \phi \frac{\partial\phi}{\partial x} - \phi \frac{\partial^2\phi}{\partial x \partial y} \right) \right\rangle - \frac{1}{2} \left\langle \frac{\partial\phi}{\partial y} \frac{\partial N\phi}{\partial x} - \phi \frac{\partial\phi}{\partial y} \frac{\partial N}{\partial x} \right\rangle$$

do $\partial/\partial y$ by parts in first piece

$$\Pi_E^x = -\frac{1}{2} \left\langle \frac{\partial N\phi}{\partial y} \frac{\partial\phi}{\partial x} - \frac{\partial N}{\partial y} \phi \frac{\partial\phi}{\partial x} \right\rangle - \frac{1}{2} \left\langle \frac{\partial\phi}{\partial y} \frac{\partial N\phi}{\partial x} - \phi \frac{\partial\phi}{\partial y} \frac{\partial N}{\partial x} \right\rangle$$

obtain two symmetric pieces

$$\Pi_E^x = \frac{1}{2} \left\langle \frac{\partial N}{\partial y} \phi \frac{\partial\phi}{\partial x} + \phi \frac{\partial\phi}{\partial y} \frac{\partial N}{\partial x} \right\rangle - \frac{1}{2} \left\langle \frac{\partial N\phi}{\partial y} \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{\partial N\phi}{\partial x} \right\rangle$$

Reynolds Stress in Symmetric Form

- E voilá: all pieces in the Reynolds stress are symmetric

$$\Pi^x = -\left\langle N\left(\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\right)\right\rangle - \left\langle \frac{\partial P_E}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial P_E}{\partial x}\right\rangle$$

has become

$$\Pi^x = \frac{1}{2}\left\langle \frac{\partial N}{\partial y}\phi\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi}{\partial y}\frac{\partial N}{\partial x}\right\rangle - \frac{1}{2}\left\langle \frac{\partial N\phi}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial N\phi}{\partial x}\right\rangle - \left\langle \frac{\partial P_E}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial P_E}{\partial x}\right\rangle$$

- due to the general nature of N and P_E, this holds for any H
  - to all orders so long as functional dependence on $\phi, |\nabla_\perp\phi|^2, \nabla_\perp^2\phi$ remains
- hence symmetry arguments apply equally to all pieces
  - symmetry breaking is dynamical and depends on details of the equilibrium
  - with zonal flow generation possibly in a small, secondary role (the P_{2E} PDF)

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