Sketch of Derivation

- cast GK eqn as generic bracket

$$\frac{\partial f}{\partial t} + \{H, f\} = C(f)$$

- small amplitude expansion

$$f = f_0 + g \qquad\qquad H = H_0 + H_1 + H_2$$

- require static background to lowest order (assume $C(f_0)$ vanishes)

$$\frac{\partial f_0}{\partial t} = 0 \implies \{H_0, f_0\} = 0 \implies f_0 = f_0(H_0)$$

- linearise polarisation:
  - ◦ H_2 terms appear only with f_0 in Lagrangian
  - ◦ **therefore** H_2 does not appear in GK eqn itself

First Order

- in bracket, g appears with H_0 and f_0 with H_1

$$\frac{\partial g}{\partial t} + \{H_0, g\} + \{H_1, f_0\} = C(g)$$

- use functional form of f_0

$$\{H_1, f_0\} = F_0' \{H_1, H_0\} \qquad \text{where} \quad F_0' \equiv -\frac{\partial f_0}{\partial H_0}$$

- resulting form

$$\frac{\partial g}{\partial t} + \{H_0, g\} - F_0' \{H_0, H_1\} = C(g)$$

  hence

$$\frac{\partial g}{\partial t} + \{H_0, h\} = C(g)$$

  where

$$h = g + F_0' H_1$$

Casimirs, conserved by collisionless part

- use bracket linearity, F any fn of H_0, integrate phase space to find

$$\sum_{\mathrm{sp}} \int d\Lambda\, F(H_0) h^\alpha \frac{\partial g}{\partial t} = -\sum_{\mathrm{sp}} \int d\Lambda\, \frac{1}{\alpha+1} \left\{ H_0, F(H_0) h^{\alpha+1} \right\} = 0$$

- special case: $F(H_0) = (F_0')^{-1}$ and $\alpha = 1$

$$\sum_{\mathrm{sp}} \int d\Lambda\, \frac{h}{F_0'} \frac{\partial g}{\partial t} = -\sum_{\mathrm{sp}} \int d\Lambda\, \frac{1}{2} \left\{ H_0, (h^2/F_0') \right\} = 0$$

- using polarisation and induction equations,

$$\sum_{\mathrm{sp}} \int d\Lambda\, \frac{h}{F_0'} \frac{\partial g}{\partial t} = \frac{\partial}{\partial t} \sum_{\mathrm{sp}} \int d\Lambda\, \frac{\partial}{\partial t} \frac{hg}{2F_0'}$$

- this defines the free energy (quad in all dep vars, consv'd by non-diss part)

$$\mathcal{E} = \sum_{\mathrm{sp}} \int d\Lambda\, \frac{hg}{2F_0'} \qquad\qquad \frac{\partial \mathcal{E}}{\partial t} = \sum_{\mathrm{sp}} \int d\Lambda\, \frac{h}{F_0'} \frac{\partial g}{\partial t} = 0$$

Polarisation Details

- polarisation comes from the terms with ϕ in the Lagrangian

$$\mathcal{L} = \cdots - \sum_{\mathrm{sp}} \int d\Lambda \left(geJ_0\phi - f_0 \frac{e^2}{2B} \frac{\partial}{\partial\mu} \left[J_0(\phi^2) - (J_0\phi)^2 \right] \right)$$

- take functional deriv, set $J_0 \to 1$ in 1st ϕ^2 term, note μB is in H_0

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \implies \sum_{\mathrm{sp}} \int dW \, \left[eJ_0 g + e^2 F_0'(J_0^2 - 1)\phi \right] = 0$$

Induction Details

- induction comes from the terms with $A_\parallel$ in the Lagrangian

$$\mathcal{L} = \cdots + \sum_{\text{sp}} \int d\Lambda \left[ge\frac{p_z}{mc} J_0 A_\parallel - f_0 \frac{e^2}{2mc^2} (J_0 A_\parallel)^2 \right] - \int dV \, \frac{1}{8\pi} \left| \nabla_\perp A_\parallel \right|^2$$

- take functional deriv

$$\frac{\delta \mathcal{L}}{\delta A_\parallel} = 0 \implies \left(k_d^2 - \nabla_\perp^2 \right) A_\parallel = \sum_{\text{sp}} \int dW \, \left[\frac{4\pi e}{mc} p_z J_0 g \right]$$

where the inverse skin depth k_d is given by

$$k_d^2 = \sum_{\text{sp}} \int dW \, \frac{4\pi e^2}{mc^2} f_0 J_0^2 = \sum_{\text{sp}} \int dW \, \frac{4\pi e^2 p_z^2}{m^2 c^2} F_0' J_0^2$$

some implications

- induction result motivates re-def of delta-f

$$g = f + F_0' \frac{e}{mc} p_z J_0 A_\parallel \qquad\qquad h = f + F_0' e J_0 \phi$$

so that Ampere's law is more evident

$$-\nabla_\perp^2 A_\parallel = \frac{4\pi}{c} \sum_{\text{sp}} \int dW \, \frac{e}{m} p_z J_0 f$$

- note polarisation is unaffected since $p_z F_0'$ is an odd function

$$\sum_{\text{sp}} \int dW \, \left[e J_0 f + e^2 F_0' (J_0^2 - 1) \phi \right] = 0$$

- linearisation of polarisation makes the identification $p_z \to m v_\parallel$ evident

Casimir Details

- examine the energy integrand $hg/2F_0'$

$$\frac{1}{2F_0'}\left[(f+eF_0'J_0\phi)\left(f+\frac{e}{mc}p_zF_0'J_0A_\parallel\right)\right]=\frac{f^2}{2F_0'}+\frac{1}{2}feJ_0\phi+\frac{1}{2}f\frac{e}{mc}p_zJ_0A_\parallel$$

- using polarisation and induction, show that

$$\sum_{\rm sp}\int d\Lambda\,\frac{1}{2}feJ_0\phi=\sum_{\rm sp}\int d\Lambda\,e^2F_0'(1-J_0^2)\frac{\phi^2}{2}\equiv\mathcal{E}_E$$

$$\sum_{\rm sp}\int d\Lambda\,\frac{1}{2}f\frac{e}{mc}p_zJ_0A_\parallel=\int dV\,\frac{1}{8\pi}\left|\nabla_\perp A_\parallel\right|^2\equiv\mathcal{E}_M$$

which are ExB and magnetic energies, respectively, and that

$$\frac{\partial\mathcal{E}_E}{\partial t}=\sum_{\rm sp}\int d\Lambda\,eJ_0\phi\frac{\partial g}{\partial t}\qquad\qquad\frac{\partial\mathcal{E}_M}{\partial t}=\sum_{\rm sp}\int d\Lambda\,f\frac{e}{mc}p_zJ_0\frac{\partial A_\parallel}{\partial t}$$

Entropy as Free Energy

- one remaining term: $f^2/2F_0'$

$$\mathcal{E}_f \equiv \sum_{\mathrm{sp}} \int d\Lambda \, \frac{f^2}{2F_0'} \qquad\qquad \frac{\partial \mathcal{E}_f}{\partial t} = \sum_{\mathrm{sp}} \int d\Lambda \, \frac{f}{2F_0'} \frac{\partial f}{\partial t}$$

- identification with entropy (Krommes 1994)

$$S = -T_0 \int d\Lambda \, f \log f \qquad\qquad \delta S = -T_0 \int d\Lambda \, \frac{(\delta f)^2}{f_0}$$

- Noether energy in total-f model

$$\mathcal{E} = \mathcal{E}_f + \mathcal{E}_E + \mathcal{E}_M \qquad \text{with} \quad \mathcal{E}_f = \sum_{\mathrm{sp}} \int d\Lambda \, H_0 f$$

- "thermal free energy" replaces $H_0 f$ in delta-f model
- energy transfer terms (pieces of $h[H_0, h]$) are the same

Nonlinearities and Energy Consistency

- $hg/2F_0'$ is the only relevant Casimir
  - (only case where entire object goes inside the bracket using field eqs)

- overriding constraint: F_0 must commute with brackets

- consequences:
  - parallel bracket is linearised
  - toroidal case must use fluxtube ordering: F_0 commutes with $\nabla_\perp$ but not $\nabla_\parallel$
  - in p_z- or $v_\parallel$-rep, therefore B can depend only on parallel space coordinate

- note that linearisation of polarisation is a separate issue
  - (can have a total-f model in which this is done)

- only way to keep nonlinear bracket is to abandon delta-f splitting

- in delta-f model the nonlinearities enter advection (drifts) only

$$\{H_1, g\} = \{H_1, h\} \qquad \textit{iff} \quad \{H_1, F_0'\} = 0$$

Delta-f Equations

$$\frac{\partial g}{\partial t} + \frac{cF^{xy}}{eB^2}[H_1, h + f_0 \log f_0]_{xy} - \frac{p_z^2/m + \mu B}{2e}\mathcal{K}(h) + \frac{B^s}{B}[H_0, h]_{zs} = C(g)$$

with

$$\sum_{\rm sp} \int dW \ \left[eJ_0 f + e^2 F_0'(J_0^2 - 1)\phi\right] = 0$$

and

$$-\nabla_\perp^2 A_\parallel = \frac{4\pi}{c} \sum_{\rm sp} \int dW \ \frac{e}{m} p_z J_0 f$$

- definitions

$$g = f + F_0' \frac{e}{mc} p_z J_0 A_\parallel \qquad\qquad h = f + F_0' e J_0 \phi$$

$$\mathcal{K} = \nabla \log R \cdot \frac{c\mathbf{F}}{B^2} \cdot \nabla_{xy} \qquad\qquad J_0\phi = \sum_{\mathbf{k}} J_0(k_\perp v_\perp/\Omega)\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$[f, g]_{xy} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \qquad \text{same for} \quad [f, g]_{zs} \quad \text{with} \quad p_z, s$$

Delta-f Forcing

- linear gradient drive (first four gyrofluid moments)

$$\nabla f_0 \log f_0 \equiv f_0 \left[\nabla \log n + \frac{p_z}{T} \nabla u_\parallel + \frac{p_z^2 - mT}{2mT} \nabla \log T_\parallel + \frac{\mu B - T}{T} \nabla \log T_\perp \right]$$

- pitch angle scattering (simple; momentum cons left to total-f)

$$C(f) = \frac{\partial}{\partial \zeta} \frac{\nu_L}{v^3} (1 - \zeta^2) \frac{\partial f}{\partial \zeta}$$

where

$$p_z = \zeta v \qquad v^2 = (p_z/m)^2 + 2\mu B \qquad \nu_L = 1.87997 \nu_{\text{Brag}}$$