Validity of the Gyrokinetic Description in the Tokamak Edge

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Summary

- **gyrokinetic theory**
  — two derivations: any-amplitude/small-rho, or, small-amplitude/any-rho
  — traditionally, these are large-scale and microturbulence ordering
  — actually, only ExB-vorticity/gyrofrequency definititively must be small

- **conventional equations**
  — derivation from Hahm, *Phys Fluids* 31 (1988) 2670
  — resulting Lagrangian is limit-taken to small $k_\perp \rho$ and small $\rho / L_\perp$

- **large-scale equations**
  — derivation from Miyato et al, *J Phys Soc Japan* 78 (2009) 104501
  — resulting Lagrangian is limit-taken to small ExB-Mach and small $L_\perp / R$

- **principle of correspondence**
  — above two limits are compared and conclusions are drawn

Hahm's gyroaveraged potential

- to lowest order we have just $\langle\phi\rangle_{\mathbf{R}}$ defined as

$$\langle\phi\rangle_{\mathbf{R}}(\mathbf{R}) = \oint \frac{d\vartheta}{2\pi}\phi(\mathbf{x})\,\delta^3(\mathbf{R}+\mathbf{a}-\mathbf{x})$$

- on a grid we Taylor expand about $\mathbf{R}$ where ∇ is with respect to $\mathbf{R}$

$$\phi(\mathbf{x}) = \phi(\mathbf{R}) + \mathbf{a}\cdot\nabla\phi + \frac{1}{2}\mathbf{aa}:\nabla\nabla\phi$$

- the gyroaverage is then

$$\langle\phi\rangle_{\mathbf{R}} = \phi(\mathbf{R}) + \frac{a^2}{4}\nabla_\perp^2\phi$$

- the gyroaverage defect and its Taylor expansion are given by

$$\widetilde{\phi} = \phi - \langle\phi\rangle_{\mathbf{R}} \qquad \widetilde{\phi} = \mathbf{a}\cdot\nabla\phi + \frac{1}{2}\mathbf{aa}:\nabla\nabla\phi - \frac{a^2}{4}\nabla_\perp^2\phi$$

Hahm's gyrocenter Lagrangian

- start with Eq. (16) TS Hahm *Phys Fluids* 31 (1988) 2670

$$L_p = \left(\frac{e}{c}\mathbf{A} + mv_\parallel \mathbf{b}\right) \cdot \dot{\mathbf{R}} + M\,\dot{\vartheta} - H \qquad\qquad H = m\frac{v_\parallel^2}{2} + M\Omega + e\Psi$$

$$f = f(\mathbf{R}, M, v_\parallel) \qquad \phi = \phi(\mathbf{x}) \to \langle\phi\rangle_{\mathbf{R}}(\mathbf{R}) \qquad \Omega = \frac{eB}{mc} \qquad M = \frac{mc}{e}\mu$$

$$\mathbf{A}^* = \frac{e}{c}\mathbf{A} + mv_\parallel \mathbf{b} \qquad\qquad \mathbf{B}^* = \nabla\times\mathbf{A}^* \qquad\qquad B_\parallel^* = \mathbf{b}\cdot\mathbf{B}^*$$

$$\Psi = \langle\phi\rangle_{\mathbf{R}} - \frac{e^2}{2\Omega}\left(\frac{\partial}{\partial M}\left\langle\widetilde{\phi}^2\right\rangle_{\mathbf{R}} + \frac{1}{m\Omega}\left\langle\nabla\widetilde{\Phi}\cdot\mathbf{b}\times\nabla\widetilde{\phi}\right\rangle_{\mathbf{R}}\right) \qquad \frac{\partial\widetilde{\Phi}}{\partial\vartheta} = \widetilde{\phi}$$

- resulting drift motion (Euler-Lagrange equations for $\mathbf{R}, v_\parallel$)

$$B_\parallel^*\dot{\mathbf{R}} = \frac{c}{e}\mathbf{b}\times\nabla H + v_\parallel\mathbf{B}^* \qquad\qquad B_\parallel^*\, m\dot{v}_\parallel = -\mathbf{B}^*\cdot\nabla H$$

long-wavelength form of potential

- use $\widetilde{\phi} = \mathbf{a} \cdot \nabla\phi$ to get these
- sense for $\mathbf{a}$

$$a^2 = \frac{2M}{m\Omega} \qquad \mathbf{a} = a\,(\cos\vartheta\,\mathbf{e}_1 - \sin\vartheta\,\mathbf{e}_2)$$

- sense of basis vectors

$$\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{b} = 1 \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{b} = \mathbf{e}_2 \cdot \mathbf{b} = 0$$

- gyroaverage defect is

$$\left\langle \widetilde{\phi}^2 \right\rangle_{\mathbf{R}} = \oint \frac{d\vartheta}{2\pi} (\mathbf{a} \cdot \nabla\phi)^2 = \frac{a^2}{2} \nabla_\perp^2 \phi$$

- ExB energy term is

$$\lim_{k_\perp \to 0} \frac{e^2}{2\Omega} \frac{\partial}{\partial M} \left\langle \widetilde{\phi}^2 \right\rangle_{\mathbf{R}} = \frac{e^2}{m\Omega^2} \left|\nabla_\perp \phi\right|^2 = \frac{mc^2}{B^2} \left|\nabla_\perp \phi\right|^2 = m\frac{u_E^2}{2}$$

long-wavelength form of drift-polarisation

- drift-polarisation potential is

$$\Psi_{22} = \frac{e^2}{2m\Omega^2} \left\langle \nabla\widetilde{\Phi} \cdot \mathbf{b}\times\nabla\widetilde{\phi} \right\rangle_{\mathbf{R}}$$

which becomes

$$\frac{e^2}{2m\Omega^2} \oint \frac{d\vartheta}{2\pi} \left(\nabla \int d\vartheta\, \mathbf{a} \cdot \nabla\phi \right) \cdot \mathbf{b}\times\nabla(\mathbf{a} \cdot \nabla\phi)$$

- expand $\mathbf{a}$ by $\sin\vartheta$ and $\cos\vartheta$ and basis vectors, do the average, find

$$\lim_{k_\perp\to 0} \Psi_{22} = \frac{e^2 a^2}{4m\Omega^2} \left[(\nabla_1\nabla_1\phi)(\nabla_2\nabla_2\phi) - (\nabla_2\nabla_1\phi)(\nabla_1\nabla_2\phi) \right]$$

where

$$\nabla_1 = \mathbf{e}_1 \cdot \nabla \qquad \nabla_2 = \mathbf{e}_2 \cdot \nabla$$

- at low-$k_\perp$ this term scales like vorticity squared and is small compared to $mu_E^2/2$

Hahm's Lagrangian to Large Scale

- in the longwave limit, the drift-polarisation potential is small
  - this is due also to $\rho \ll L_\perp$

- the rest of the Lagrangian expanded to low-$k_\perp \rho \ll 1$ is

$$L_p = \left(\frac{e}{c} \mathbf{A} + m v_\parallel \mathbf{b} \right) \cdot \dot{\mathbf{R}} + M \, \dot{\vartheta} - H$$

$$H = m \frac{v_\parallel^2}{2} + M\Omega + e\phi + e \frac{a^2}{4} \nabla_\perp^2 \phi - m \frac{u_E^2}{2}$$

where

$$\Omega = \frac{eB}{mc} \qquad a^2 = \frac{2M}{m\Omega} \qquad u_E = \frac{c}{B} \left| \nabla_\perp \phi \right|$$

- for electromagnetic cases add appropriate field term and replace

$$m v_\parallel \mathbf{b} \to p_z \mathbf{b} \qquad m \frac{v_\parallel^2}{2} \to \frac{1}{2m} \left(p_z - \frac{e}{c} A_\parallel \right)^2 \qquad p_z = \frac{e}{c} A_\parallel + m v_\parallel$$

Our finite-ExB-Mach Lagrangian

- "Ours" means what we derived in
  Miyato, Scott, Tokuda, Strintzi, *J Phys Soc Japan* 78 (2009) 104501

- flows considered with maximal ordering versus thermal energy and $L_\perp \sim R$

$$L_p = \left(\frac{e}{c}\mathbf{A} + m v_\parallel \mathbf{b}\right) \cdot \dot{\mathbf{R}} + M\,\dot{\vartheta} - H$$

where

$$H = m\frac{v_\parallel^2}{2} + M\Omega + e\phi - m\frac{u_E^2}{2} + H_2 \qquad \Omega = \frac{eB}{mc} \qquad u_E = \frac{c}{B}\left|\nabla_\perp \phi\right|$$

and the 2nd-order H is (Eq. 46 of our paper, in its notation)

$$\begin{aligned} H_2 = {} & \frac{mc}{2e}\left(\mu + \frac{m u_E^2}{2B}\right)\mathbf{b}\cdot\nabla\times\mathbf{D} - \frac{m v_\parallel}{2\Omega}\mathbf{D}\cdot\nabla\times\mathbf{D} \\ & + \frac{m}{2\Omega}\mathbf{b}\times\mathbf{D}\cdot\frac{\partial \mathbf{D}}{\partial t} + \frac{7}{6}\frac{\mu B}{\Omega}\mathbf{b}\times\mathbf{D}\cdot\nabla\log B \\ & + \frac{m v_\parallel}{2\Omega}u_E^2\mathbf{b}\cdot\nabla\times\mathbf{b} - \frac{(m v_\parallel)^2}{\Omega^2}e\nabla\phi\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \end{aligned}$$

$$\mu B = M\Omega$$

$$\mathbf{D} = \frac{c}{B}\mathbf{b}\times\nabla\phi$$

Our Lagrangian under Edge Conditions

- in the edge $L_\perp/R \sim \rho/L_\perp \ll 1$ so only the first term in H_2 remains

$$H_2 = \frac{mc}{2e}\left(\mu + \frac{mu_E^2}{2B}\right)\mathbf{b}\cdot\nabla\times\mathbf{D}$$

and since $L_\perp \ll R$

$$H_2 = e\frac{2M\Omega + mu_E^2}{4m\Omega^2}\nabla_\perp^2\phi$$

- in the edge we also have $u_E/c_s \sim \rho/L_\perp \ll 1$
  - the result is that while u_E^2 is kept at O(1) it is small in H_2 which leaves

$$H = m\frac{v_\parallel^2}{2} + M\Omega + e\phi + e\frac{a^2}{4}\nabla_\perp^2\phi - m\frac{u_E^2}{2} \qquad a^2 = \frac{2M}{m\Omega}$$

- therefore our Lagrangian is the same as Hahm's under these conditions at large scale

Hahm's Lagrangian covers the whole spectrum without difficulty

measuring the finite ExB-Mach effects

- in our Lagrangian we keep the shear-Alfvén and ExB-Mach effects . . .
  - where $\mathcal{L}$ is the entire Lagrangian density including the magnetic field term $\mathcal{L}_f$

$$L_p = \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b}\right)\cdot\dot{\mathbf{R}} + M\,\dot{\vartheta} - H \qquad\qquad \mathcal{L}_f = -\frac{1}{8\pi R^2}\left|\nabla_\perp\left(\psi + A_\parallel R\right)\right|^2$$

$$H = \frac{1}{2m}\left(p_z - \frac{e}{c}A_\parallel\right)^2 + M\Omega + e\phi - m\frac{u_E^2}{2} + e\frac{a^2}{4}\nabla_\perp^2\phi \qquad a^2 = \frac{2M\Omega + mu_E^2}{m\Omega^2}$$

- . . . and we consider the flux terms in toroidal angular momentum conservation

$$\frac{\partial}{\partial t}\left\langle\sum_{\mathrm{sp}}\int d\mathcal{W}\,fp_z b_\varphi - \mathbf{P}\cdot\nabla\frac{1}{c}\psi\right\rangle + \frac{\partial}{\partial V}\left\langle\sum_{\mathrm{sp}}\int d\mathcal{W}\,fp_z b_\varphi\dot{\mathbf{R}}\cdot\nabla V\right\rangle$$

$$+\frac{\partial}{\partial V}\left\langle\left[\left(-\frac{\partial\mathcal{L}}{\partial\nabla_\perp\phi}\frac{\partial\phi}{\partial\varphi}\right) + \left(-\frac{\partial\mathcal{L}}{\partial\nabla_\perp A_\parallel}\frac{\partial A_\parallel}{\partial\varphi}\right)\right]\cdot\nabla V\right\rangle$$

$$+\frac{\partial}{\partial V}\left\langle\left[\left(\frac{\partial}{\partial\varphi}\frac{\partial\mathcal{L}}{\partial\nabla_\perp^2\phi}\right)\nabla_\perp\phi + \frac{\partial\phi}{\partial\varphi}\left(\nabla_\perp\frac{\partial\mathcal{L}}{\partial\nabla_\perp^2\phi}\right)\right]\cdot\nabla V\right\rangle = 0$$

contributions to polarisation/ExB momentum

- with the FLR contribution a^2 and its correction mu_E^2
  ◦ we keep single species ions and assume $m_e \ll m_i$ so only ion inertia enters

$$-\mathbf{P} \cdot \frac{1}{c}\nabla\psi = \left[\left(1 - \frac{\Omega_E}{2\Omega_i}\right) \frac{n m_i c^2}{B^2} \nabla_\perp \phi + \frac{Ze}{m_i} \nabla_\perp \frac{2p_\perp + n m_i u_E^2}{4\Omega_i^2} \right] \cdot \frac{1}{c}\nabla\psi$$

where

$$n = \int d\mathcal{W}\, f \qquad p_\perp = \int d\mathcal{W}\, f\, M\Omega_i \qquad \Omega_i = \frac{m_i c}{ZeB} \qquad \Omega_E = \frac{c}{B}\nabla_\perp^2 \phi$$

- the Ω_E/Ω_i term is very small (rho-star squared)

- the ExB Mach correction to $p_\perp$ enters in the FLR term (adds at most a few percent)

- the polarisation effects are similar (fs-avg dominates any coefficients)
  see also Y Idomura *Comput Sci Discov* 5 (2012) 014018

contributions to the drift momentum flux

- the gyrocenter radial drifts are given through gradients of H

$$B_\parallel^* \dot{\mathbf{R}} \cdot \nabla V = \frac{c}{e} \nabla V \cdot \mathbf{b} \times \nabla H$$

- split $H_{1,2,3}$ according to powers of ϕ, where $\rho_L^2 = 2M/m_i\Omega_i$ and $\rho_E = u_E/\Omega_i$

$$H_1 = \left(1 - \frac{\rho_L^2}{4} \nabla_\perp^2\right) e\phi \qquad H_2 = -m_i \frac{u_E^2}{2} \qquad H_3 = -\frac{\rho_E^2}{4} \nabla_\perp^2 e\phi$$

- the drift fluxes are given through moments of these over $f p_z b_\varphi$

- the H_2 and H_3 contributions to these are always very small
  ◦ in H_1 the FLR piece gives diamagnetic vorticity effects in the turbulence

- hence most of my effort has been on the wave fluxes
  (PPPL seminars 2011-13, EU-TTF 2012, EFTC 2013, APS 2012/4)

contributions to the wave momentum flux

- with $H_{1,2,3}$ given according to powers of ϕ . . .

$$H_1 = \left(1 - \frac{\rho_L^2}{4}\nabla_\perp^2\right) e\phi \qquad H_2 = -m_i \frac{u_E^2}{2} \qquad H_3 = -\frac{\rho_E^2}{4}\nabla_\perp^2 e\phi$$

- . . . the wave fluxes are given through dependence on u_E^2 or Ω_E

$$N_E = \frac{n m_i c^2}{B^2}\left(1 - \frac{\Omega_E}{2\Omega_i}\right) \qquad P_E = \frac{Ze}{m_i}\frac{2p_\perp + n m_i u_E^2}{4\Omega_i^2}$$

- the ExB wave flux is (here, ρ is a toroidal flux radius in meters)

$$\Pi_E^\rho = \left\langle -\frac{\partial\phi}{\partial\varphi}\left(N_E\nabla_\perp\phi + \nabla_\perp P_E\right)\cdot\nabla\rho - \frac{\partial P_E}{\partial\varphi}\nabla_\perp\phi\cdot\nabla\rho\right\rangle$$

- all effects from ρ_E^2 giving rise to H_3 are found to be small (shown below)

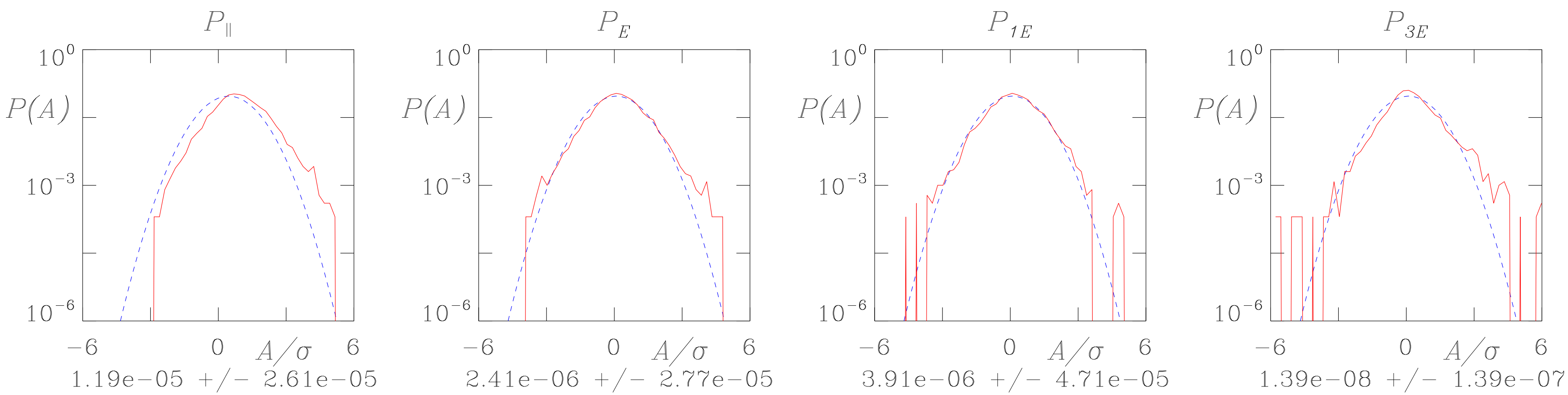
results in global edge turbulence

- from the GEMR global delta-f gyrofluid model (PoP 10/2010, see references)
- edge L-mode base case (B Scott, Contrib Plasma Phys 46:714 2006)

$$T_e = T_i = 100\,\mathrm{eV} \qquad n_e = n_i = 2.0 \times 10^{13}\,\mathrm{cm}^{-3} \qquad B = 2.5\,\mathrm{T}$$

$$R = 165\,\mathrm{cm} \qquad L_T = L_\perp = 3.5\,\mathrm{cm} \qquad L_n = 7.0\,\mathrm{cm} \qquad q = 3.5 \qquad \hat{s} = 1.14$$

- normalised parameters $\hat{\beta} = 1.75, \quad \hat{\mu} = 7.41, \quad C = 3.11, \quad \nu_B = 0.13$
  (B Scott Phys Plasmas 6/2005, PPCF 12/2003 and 12/2006)
- flux distributions for main drift flux, ExB and FLR Reynolds wave fluxes
  and third order ExB Mach corrections



Conclusion

- **principle of correspondence**
  — the conventional limit of both Lagrangians are found to match

- **resulting implication**
  — so long as ExB-Mach and $\rho/L_\perp$ and $L_\perp/R$ are indeed small,
  the conventional Lagrangian covers the large-scale large-amplitude one
  — therefore, the conventional model is better than its derivation,
  and it can be used without modification for the entire spectrum

- **tokamak edge dynamics is electromagnetic**
  — but the finite-beta Lagrangian of Hahm et al *PF* 31 (1988) 1940 covers it
  — usually used is a form with $\beta_e \ll 1$ and $\mu_e \ll 1$
  where $A_\|$ is not gyroaveraged and appears only in H and the field term

these limits hold in conventional ("ITER-like") tokamaks
no barrier exists against using gyrokinetics in the tokamak edge