

Theory and Computation in Full-F Gyrokinetics

B. Scott

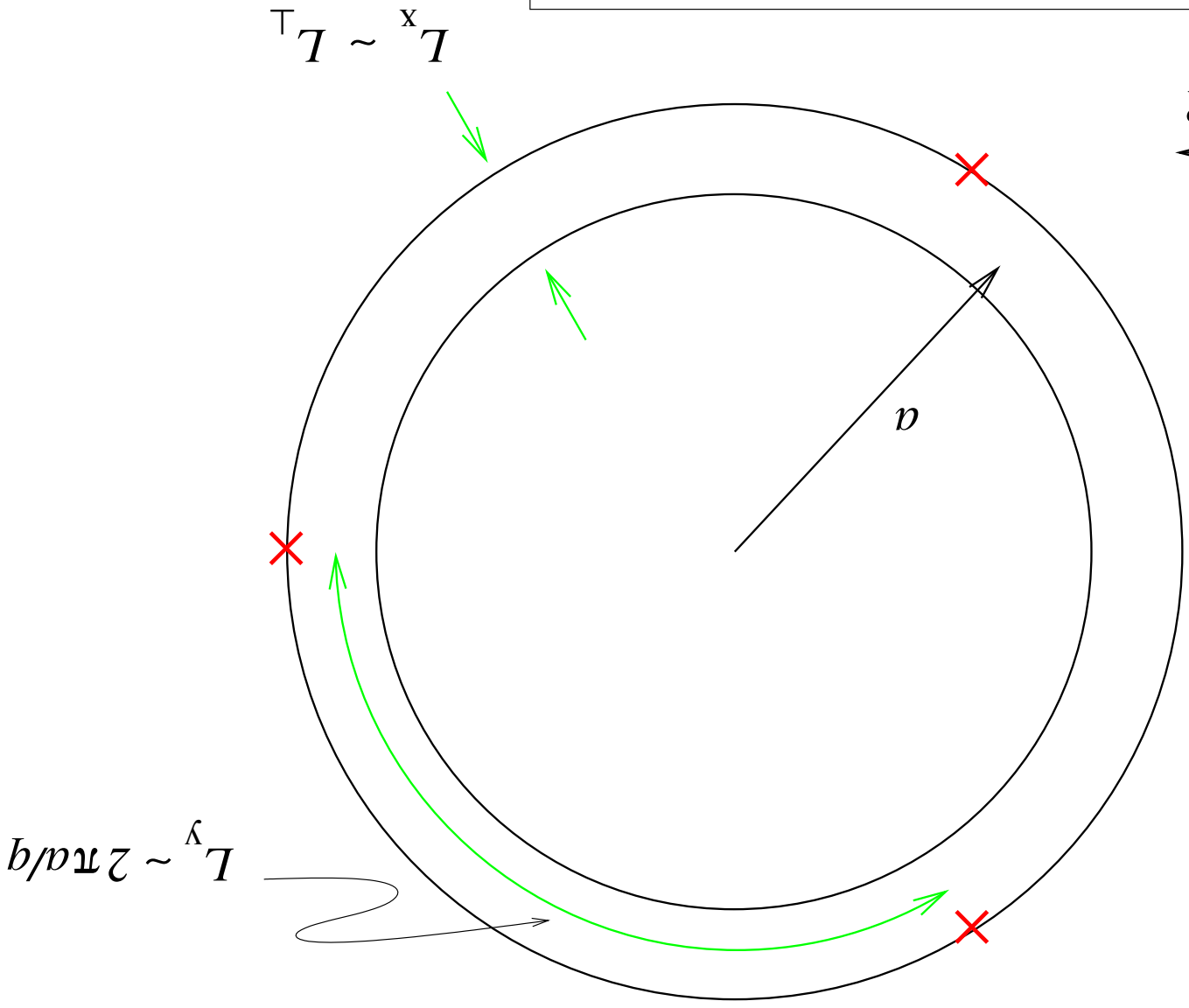
Max Planck Institut für Plasmaphysik
Euratom Association
D-85748 Garching, Germany
Theory Ringberg, 8-12 Nov 2004



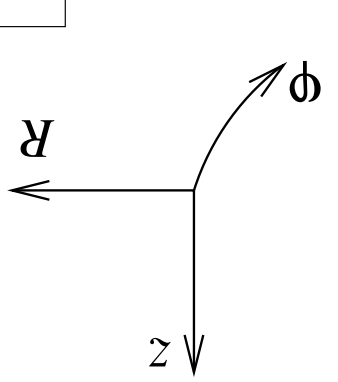
Outline

- **Preliminaries**
 - the edge situation
 - basic ideas for fully consistent gyrokinetics
- **What Gyrokinetics Means, Gyrokinetic Equation**
 - electromagnetic, fully nonlinear
 - Lagrangian, Hamiltonian structure
 - variational principle \rightarrow polarisation
- **Energy Conservation**
 - particle versus field Hamiltonian
 - drift energy versus thermal energy
 - polarisation equation and energy theorem
 - phase space conservation
- **The FEI Code**
 - implementation of above ideas
 - energy conservation, collision model
 - kinetic shear Alfvén results

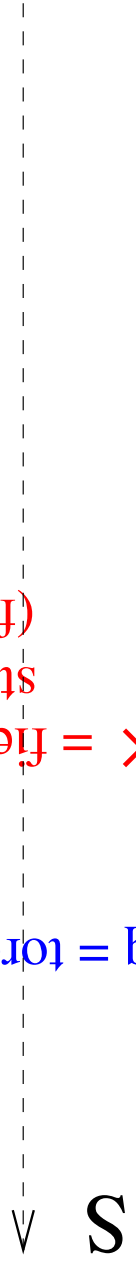
Computational Domain



large aspect ratio: $L_y \gg L_x$



$q = \text{toroidal turns/}$
 poloidal turns
 $\times = \text{field line}$
 strike points
 $(\text{for } q = 3)$



edge parameters

- typical situation: Alfvén/electron transit, collision, and drift frequencies comparable
- drift frequency is c_s/L_\perp , spectral range of main interest is $0.1 < k_y \rho_s < 1$

- steep gradient

$$\hat{\mu} \equiv \frac{m_e}{M_D} \left(\frac{L_\perp}{qR} \right)^2 = \left(\frac{c_s/L_\perp}{V_e/qR} \right)^2 < 1$$

- collisional

$$C \equiv \frac{0.51 \nu_e m_e}{c_s/L_\perp M_D} \left(\frac{L_\perp}{qR} \right)^2 = 0.51 \frac{\nu_e c_s/L_\perp}{V_e/qR} > 1$$

- electromagnetic

$$\hat{\beta} \equiv \frac{4\pi p_e}{B^2} \left(\frac{L_\perp}{qR} \right)^2 = \left(\frac{c_s/L_\perp}{v_A/qR} \right)^2 \gtrsim 1$$

Basic Ideas

- particle Lagrangian L_p
- continuum: distribution function times L_p , plus field contributions
- incorporate fields \rightarrow full self consistency
- Noether theorem \rightarrow conservation laws for *this* system

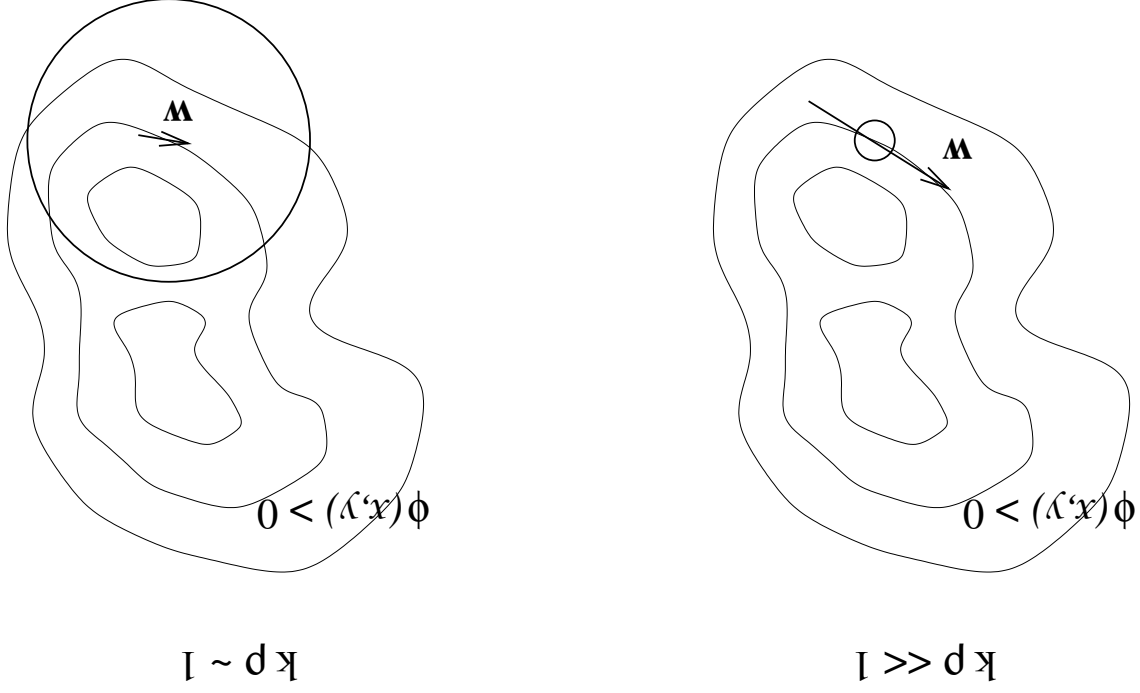
Why Continuum?

- “Vlasov model”
- usual blah blah about noise not the real reason . . .
- with 20 years fluid experience, it is what I know
 - and is now becoming tractable
- same basic methods as mature GEM (gyrofluid electromagnetic) code

The Meaning of Gyrokinetics

- low frequencies $\omega \ll \Omega_c = eB/mc$ for each species

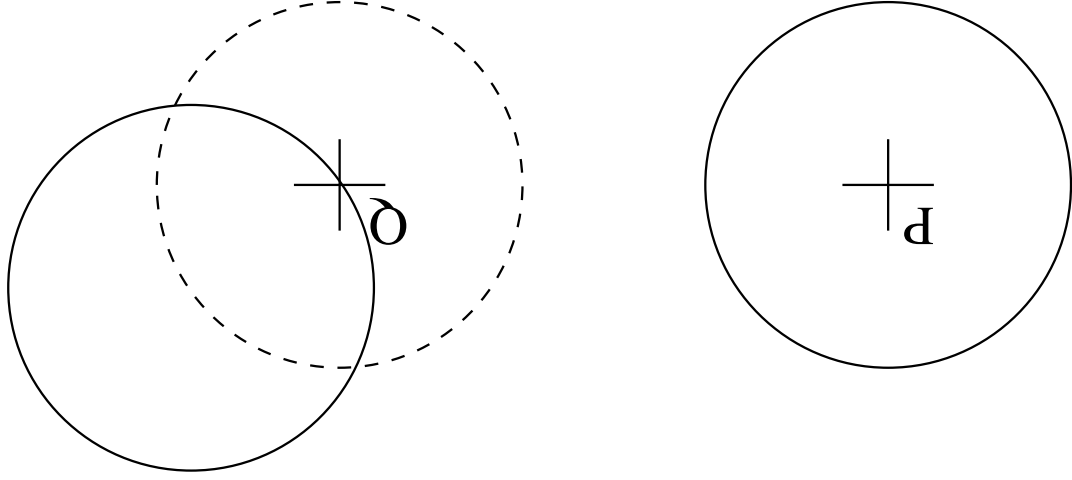
treat particles as rings of charge in spatially varying fields



- reduced response: “gyroaveraging”

- reaction to fields, polarisation density: “gyroscreening”

Gyrokinetic/Gyrofluid Polarisation



gyrocenter at P senses potential at points on the ring

gyrocenters on dashed ring contribute charge to Q

mathematics identical in both procedures (Hermitian operators)

polarisation density \propto part of f dependent upon gyrophase angle

Basic Procedure

Guiding Center Coordinates

- phase space extension of original EM Lagrangian \rightarrow "fundamental one form"

$$L = m \frac{|\dot{\mathbf{x}}|^2}{2} + \frac{e}{c} \mathbf{A} \cdot \dot{\mathbf{x}} - e\phi$$

becomes

$$\gamma = \left(\frac{e}{c} \mathbf{A} + m\mathbf{v} \right) \cdot \mathbf{dx} - H dt$$

with

$$H = m \frac{v^2}{2} + e\phi$$

- change to guiding center coordinates for motion in background magnetic field

$$\gamma(\mathbf{X}, U, \mu, \theta) = \left(\frac{e}{c} \mathbf{A} + mU\mathbf{b} \right) \cdot \mathbf{dR} + \mu B \frac{d\Omega}{d\theta} - H_0 dt$$

where θ is gyrophase angle, $\mathbf{R} = \mathbf{x} - \rho_L \mathbf{a}$ is guiding center position, and

$$H_0 = m \frac{U^2}{2} + \mu B \quad \mu = \frac{m v_{\perp}^2}{2B} \quad U = v_{\parallel}$$

- ϕ and A_{\parallel} are the dynamical fields to be added later

Lie Transform to Gyrocenter Coordinates

- Lie transform, counts fields as potential energy, part of Hamiltonian

$$\Gamma_{(0,1)} = \left[\frac{e}{c} (\mathbf{A} + \langle A_{\parallel} \rangle \mathbf{b}) + mU\mathbf{b} \right] \cdot d\mathbf{R} + \mu_B \frac{\Omega}{d\theta} - H_{(0,1)} dt$$

where

$$H_{(0,1)} = m \frac{U^2}{2} + \mu_B + e \langle \phi \rangle + \frac{e^2}{2} \frac{\partial}{\partial \mu} (\phi - \langle \phi \rangle)^2$$

- Euler Lagrange equations for gyrocenters

$$\dot{\mathbf{R}} = \mathbf{b}_* \left(\frac{1}{m} \frac{\partial U}{\partial H} \right) - \frac{e}{c} \mathbf{B} B_*^{\parallel} \cdot \nabla H$$

$$\dot{U} = -\frac{1}{m} \mathbf{b}_* \cdot \nabla H - \frac{e}{mc} \frac{\partial \langle A_{\parallel} \rangle}{\partial t}$$

where

$$\mathbf{B}_* = \nabla \times \left[(\mathbf{A} + \langle A_{\parallel} \rangle \mathbf{b}) + \frac{e}{mc} U \mathbf{b} \right]$$

$$B_*^{\parallel} = \mathbf{b} \cdot \mathbf{B}_*$$

$$\mathbf{b}_* = \mathbf{B}_* / B_*^{\parallel}$$

with gyromotion

$$\dot{\theta} = \frac{\Omega}{H} \frac{\partial B}{\partial \mu} = \Omega$$

$$\dot{\mu} = -\frac{\Omega}{H} \frac{\partial B}{\partial \theta} = 0$$

make approximations in Lagrangian rest is rigorously consistent

The Gyrokinetic Model

- electromagnetic, full-f
- particle Lagrangian

$$L_p = \left[\frac{c}{e} (\mathbf{A} + A_{\parallel} \mathbf{b}) + m v_{\parallel} \mathbf{b} \right] \cdot \dot{\mathbf{R}} + \mu \frac{\Omega}{B} \theta - \left(m \frac{v_{\parallel}^2}{2} + \mu B - e\psi \right)$$

- Hamiltonian

$$H = m \frac{v_{\parallel}^2}{2} + \mu B - e\psi$$

- gyroaveraged and (low- k_{\perp}) screened electrostatic potential

$$e\psi = e J_0 \phi - m \frac{v_E^2}{2}$$

- modified field

$$\mathbf{A}^* = \mathbf{A} + A_{\parallel} \mathbf{b} + \frac{e}{mc} v_{\parallel} \mathbf{b}$$

$$\mathbf{B}^* = \nabla \times \mathbf{A}^*$$

$$B_{\parallel}^* = \mathbf{B}^* \cdot \mathbf{b}$$

$$\mathbf{b}^* = \frac{\mathbf{B}^*}{B_{\parallel}^*}$$

Some Notes

- A_{\parallel} is not gyroaveraged: belongs to electrons
- range of scales: profile scale L_{\perp} down to drift scale ρ_s , comparable to ion gyroradius ($T_i \sim T_e$)
- low- k_{\perp} form for screening: important mostly at large scale, where strongest velocities are
- forms motivated by computational tractability

The Gyrokinetic Equation

- particle Lagrangian

$$L_p = \frac{e}{c} \mathbf{A}_* \cdot \dot{\mathbf{R}} + \mu \frac{\Omega}{B} \dot{\theta} - H$$

- equation of motion for distribution function

$$\frac{\partial f}{\partial t} - \frac{e}{m} \frac{\partial f}{\partial v_{\parallel}} \frac{c}{\partial t} \frac{1}{\partial A_{\parallel}} = \frac{1}{m} \left(\frac{\partial f}{\partial v_{\parallel}} \mathbf{p}_* \cdot \nabla H - \frac{\partial H}{\partial v_{\parallel}} \mathbf{p}_* \cdot \nabla f \right) + \frac{e B B_{\parallel}^*}{c} \mathbf{F} : (\nabla H \nabla f)$$

- tensor representation for magnetic field

$$\mathbf{F} = \epsilon \cdot \mathbf{B} \quad \text{hence} \quad \mathbf{v}_E = \frac{B_z}{c} \mathbf{B} \times \nabla \phi = -\frac{B_z}{c} \mathbf{F} \cdot \nabla \phi$$

- Hamiltonian structure

$$\frac{\partial f}{\partial t} - \frac{e}{m} \frac{\partial f}{\partial v_{\parallel}} \frac{c}{\partial t} \frac{1}{\partial A_{\parallel}} = [H, f]$$

where, as we will see, the term $\partial A_{\parallel} / \partial t$ tracks magnetic energy,

and where ...

... where we have a spatial coordinate system $\{x, y, s\}$ so that

$$[H, f] = \frac{m}{b_{*x}} \left(\frac{\partial f}{\partial H} \frac{\partial v_{\parallel}}{\partial x} - \frac{\partial v_{\parallel}}{\partial H} \frac{\partial f}{\partial x} \right) + \frac{e B B_{*}^{\parallel}}{c F_{xy}} \left(\frac{\partial f}{\partial H} \frac{\partial x}{\partial y} - \frac{\partial x}{\partial H} \frac{\partial f}{\partial y} \right)$$

$$+ \frac{m}{b_{*y}} \left(\frac{\partial f}{\partial H} \frac{\partial v_{\parallel}}{\partial y} - \frac{\partial v_{\parallel}}{\partial H} \frac{\partial f}{\partial y} \right) + \frac{e B B_{*}^{\parallel}}{c F_{xs}} \left(\frac{\partial f}{\partial H} \frac{\partial x}{\partial s} - \frac{\partial x}{\partial H} \frac{\partial f}{\partial s} \right)$$

$$+ \frac{m}{b_{*s}} \left(\frac{\partial f}{\partial H} \frac{\partial v_{\parallel}}{\partial s} - \frac{\partial v_{\parallel}}{\partial H} \frac{\partial f}{\partial s} \right) + \frac{e B B_{*}^{\parallel}}{c F_{ys}} \left(\frac{\partial f}{\partial H} \frac{\partial y}{\partial s} - \frac{\partial y}{\partial H} \frac{\partial f}{\partial s} \right)$$

• notes:

- curvature drift is in the b_{*} terms
- grad-B drift is in the μB part of the drift terms
- (generalised) ExB drift is in the $e\psi$ part of the drift terms
- parallel dynamics is in the b terms

• note especially:

- parallel dynamics and trapping go together, in both $e\psi$ and μB
- (really subtle) phase conservation:

$$\frac{e}{\partial} \frac{m \partial v_{\parallel}}{\partial} (\mathbf{b}_{*} \cdot \nabla \psi) \quad \text{versus} \quad \nabla \cdot \frac{B B_{*}^{\parallel}}{c} \mathbf{F} \cdot \nabla \psi$$

◦ that is, curvature drift part of trapping versus ExB advection!

Polarisation: Reaction of the Field

- find from Hamilton's principle, variation of ϕ

$$\sum^{\text{sp}} \left[\int dW e^{J_0} f + \Delta \cdot \frac{Bz}{mc^2} \Delta^\perp \phi \right] = 0 \quad \text{where} \quad n = \int dW f$$

- a little slower ...

◦ position gyrocenters in space (G is positioning kernel), don't forget sum over species:

$$\int dV \mathcal{L} = \sum^{\text{sp}} \int dV L^p G(\mathbf{R}, \mathbf{x})$$

◦ examine part due to the potential ϕ :

$$\int dV \mathcal{L} = \sum^{\text{sp}} \int dV \int dW f(\dots) = \sum^{\text{sp}} \int dV \int dW f \left(e^{J_0} \phi - \frac{2Bz}{mc^2} |\Delta^\perp \phi|^2 \right)$$

◦ vary with respect to ϕ (J_0 is Hermitian, do one Δ^\perp by parts, incorporate f into n):

$$\delta \left(\int dV \mathcal{L} \right) = - \sum^{\text{sp}} \int dV \delta \phi \left(\int dW e^{J_0} f + \Delta \cdot \frac{Bz}{mc^2} \Delta^\perp \phi \right)$$

Polarisation (2): Ampere's Law

- find from Hamilton's principle, variation of A_{\parallel}

$$\int dV \mathcal{L} = \dots + \sum_{\text{sp}} \int dV \int dW f \left(\frac{c}{e} A_{\parallel} v_{\parallel} \right) - \int dV \frac{1}{8\pi} |\Delta_{\perp} A_{\parallel}|^2$$

- vary with respect to A_{\parallel} (do one Δ_{\perp} by parts):

$$\delta \left(\int dV \mathcal{L} \right) = \int dV \delta A_{\parallel} \left[\sum_{\text{sp}} \int dW \frac{c}{e} v_{\parallel} f \right] + \frac{1}{4\pi} \Delta_{\perp}^{\perp} A_{\parallel}$$

- resulting Ampere's law

$$0 = f v_{\parallel} \int dW \frac{c}{4\pi} \sum_{\text{sp}} + \Delta_{\perp}^{\perp} A_{\parallel}$$

closed system: gyrokinetic equation for each f and polarisation equations for both ϕ and A_{\parallel}

Energy Conservation

- main point:

particle and field energy work together

- in this case, field energy controlling ϕ is actually ExB motion of particles

- an important auxiliary relation using polarisation:

$$\sum_{\text{ds}} \int dV \int dW e J_0 \phi f = \sum_{\text{ds}} \int dV m v \frac{E}{2}$$

- proof: multiply polarisation by ϕ , integrate over volume

$$0 = \left[\int dV \phi \sum_{\text{ds}} \left[\int dW e J_0 f + \Delta \cdot \frac{B_2}{m c^2} \Delta^\perp \phi \right] \right]$$

$$\int dV \sum_{\text{ds}} \left[\int dW \phi e J_0 f - \Delta \phi \cdot \frac{B_2}{m c^2} \Delta^\perp \phi \right]$$

$$\int dV \sum_{\text{ds}} \int dW f e J_0 \phi = \int dV \sum_{\text{ds}} \frac{B_2}{m c^2} |\Delta^\perp \phi|^2$$

ExB Energy

- “charge density” is really ExB energy
 - start with

$$\sum^{\text{sp}} \int dV \int dW e J_0 \phi f = \sum^{\text{sp}} \int dV m m v_E^2$$

- subtract the ExB energy from both sides

$$\sum^{\text{sp}} \int dV \left(\int dW e J_0 \phi f - m m \frac{v_E^2}{2} \right) = \sum^{\text{sp}} \int dV m m \frac{v_E^2}{2} \equiv \mathcal{E}_E$$

- combine left side into effective potential, move to right side

$$\mathcal{E}_E = \sum^{\text{sp}} \int dV \int dW e \phi f$$

ExB energy is equivalent to
total gyrocenter potential energy

Magnetic Energy

- standard form in terms of magnetic field (integration of one Δ_{\perp} by parts)

$$\mathcal{E}_M = \int dV \frac{1}{8\pi} |\Delta_{\perp} A_{\parallel}|^2 = - \int dV \frac{1}{8\pi} (\Delta_{\perp}^{\perp} A_{\parallel}) A_{\parallel}$$

- insert form from Ampere's law

$$\mathcal{E}_M = \sum_{\text{sp}} \int dV \frac{1}{2} \int dW e v_{\parallel} f \frac{c}{1} A_{\parallel}$$

- note that $v_{\parallel} = (1/m) \partial H / \partial z$, and do that derivative by parts (note this H includes $e\psi$)

$$\mathcal{E}_M = - \sum_{\text{sp}} \int dV \int dW H \frac{1}{2m} \frac{\partial v_{\parallel}}{\partial f} \frac{c}{e} A_{\parallel}$$

magnetic energy is equivalent to acceleration
by force potential in magnetic field

Thermal Energy

- this is simply the unperturbed Hamiltonian weighted by the particles

$$\mathcal{E}_H = \int dV \int dV \sum_{\text{sp}} f H_0$$

$$H_0 = \frac{mv_{\parallel}^2}{2} + \mu B$$

where

Total Energy

- Hamiltonian energy is therefore thermal plus ExB energy

$$\mathcal{E}_F + \mathcal{E}_E = \sum_{\text{sp}} \int dV \int dW H f = \sum_{\text{sp}} \int dV \int dW (H_0 + e\psi) f$$

- this adds to magnetic energy

$$\mathcal{E} = \mathcal{E}_F + \mathcal{E}_E + \mathcal{E}_M = \sum_{\text{sp}} \int dV \int dW H \left[f - \frac{1}{e} \frac{\partial f}{\partial v_{\parallel}} \frac{zm}{e} v_{\parallel} A_{\parallel} \right]$$

- part of energy due to H_0 is thermal energy

- part of energy due to $e\psi$ is ExB energy (“drift energy” or “electrostatic field energy”)

- part of energy due to $(e/c)A_{\parallel}$ is magnetic energy

Energy Conservation

- time derivatives of field energies work the same way (more manipulations):

$$\frac{\partial \mathcal{E}_E}{\partial t} = \frac{\partial}{\partial t} \int dV \sum_{\text{ds}} \frac{v_E^2}{2} = \int dV \sum_{\text{ds}} \frac{\partial}{\partial t} \phi_e$$

$$\frac{\partial \mathcal{E}_M}{\partial t} = \frac{\partial}{\partial t} \int dV \frac{1}{8\pi} |\Delta_{\perp} A_{\parallel}|^2 - \int dV \sum_{\text{ds}} H \frac{m}{e} \frac{\partial v_{\parallel}}{\partial t} \frac{\partial \phi}{\partial t}$$

- so that the evolution of the total energy is given by ...

$$\frac{\partial \mathcal{E}}{\partial t} = \int dV \sum_{\text{ds}} \left[\frac{\partial}{\partial t} (H_0 + e\psi) - \frac{\partial}{\partial t} H \frac{m}{e} \frac{\partial v_{\parallel}}{\partial t} \frac{\partial \phi}{\partial t} \right]$$

$$= \int dV \sum_{\text{ds}} \left[\frac{\partial f}{\partial t} - \frac{m}{e} \frac{\partial v_{\parallel}}{\partial t} \frac{\partial \phi}{\partial t} \right] H$$

$$= \int dV \sum_{\text{ds}} [f, H] = 0$$

... zero! One more little detail however ...

Phase Space Conservation

... the assertion that the integral over $H[H, f]$ vanishes requires the Liouville theorem

• express the bracket in advection form

$$[H, f] = -V^\alpha \Delta^\alpha f \quad \text{where} \quad V^\alpha = \left\{ -\frac{eBB^*}{c} \mathbf{F} \cdot \Delta H + \frac{m}{H} \frac{\partial H}{\partial \mathbf{p}^*}, -\frac{m}{1} \mathbf{p}^* \cdot \Delta H \right\}$$

• it is important to note that not $\Delta^\alpha V^\alpha$ but $\Delta^\alpha (B^* V^\alpha)$ is zero
 ◦ proof (use $\Delta \times \mathbf{b} = -\Delta \cdot (\mathbf{F}/B)$ and $\Delta \cdot \mathbf{B}^* = 0$):

$$\Delta^\alpha (B^* V^\alpha) = -\Delta \cdot \left(\frac{c}{\mathbf{F}} \frac{e}{B} \cdot \Delta H \right) - \frac{1}{\partial \mathbf{B}^*} \frac{m}{\partial v_{\parallel}} \cdot \Delta H$$

$$= -\Delta \cdot \left(\frac{c}{\mathbf{F}} \frac{e}{B} \cdot \Delta H \right) - \frac{e}{c} (\Delta \times \mathbf{b}) \cdot \Delta H$$

$$= -\Delta \cdot \left(\frac{c}{\mathbf{F}} \frac{e}{B} \cdot \Delta H \right) + \frac{e}{c} \left(\Delta \cdot \frac{\mathbf{B}}{\mathbf{F}} \right) \cdot \Delta H$$

$$= 0$$

• the phase space element is $\int dW = 2\pi \int dv_{\parallel} d\mu B^*$

Phase Space Conservation (2)

- thus we have

$$H[H, f] = \left[\frac{H^2}{2}, f \right] = -\Delta_{V^\alpha}^2 f$$

where V_2 is V defined with $H \mapsto H^2/2$

- number and energy conservation follow the same way:

$$\int dV \int dW [H, f] = -\Delta_{V^\alpha}^2 f$$

$$= -\Delta_{V^\alpha}^2 f$$

that is, zero.

- find energy conservation using $H \mapsto H^2/2$ and $V \mapsto V_2$ (trivial)

- find entropy conservation using $(\log f + 1)[H, f] = [f \log f]$

- note that the $(\partial f / \partial z)(\partial A_{\parallel} / \partial t)$ term conserves number and entropy trivially

Energy Transfer

- using total energy conservation it is simple to get the transfer, e.g., from thermal to ExB and magnetic energy, respectively:

$$\frac{\partial \mathcal{E}_F}{\partial t} = \frac{\partial \mathcal{E}_E}{\partial t} - \frac{\partial \mathcal{E}_M}{\partial t}$$

$$\sum^{\text{sp}} \int dV \int dW \frac{\partial f}{\partial t} H_0 = - \sum^{\text{sp}} \int dV \int dW \left[e\psi \frac{\partial f}{\partial t} - ev_{\parallel} f \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right]$$

- the energy theorem for zonal flows is found from the zonal contribution to ExB energy:

- define the zonal average $\langle \phi \rangle$ according to coordinate system
- field aligned, unit Jacobian Hamada coordinates x, y, s : $\langle \phi \rangle = \int dy \int ds \phi$

$$\sum^{\text{sp}} \int dx \int dW \langle e\psi \rangle \frac{\partial \langle f \rangle}{\partial t} = \langle \text{“Reynolds stress”} + \text{“Maxwell stress”} - \text{“geodesic transfer”} \rangle$$

- from the ExB and magnetic nonlinearities and curv/grad-B drifts, respectively

- gyrokinetic codes should be using relations like these as diagnostics!

The FEFI Model

- no delta-f approximations, Lagrangian is

$$L = \sum^{\text{sp}} \int dV f \left[\frac{e}{c} \mathbf{A}^* \cdot \dot{\mathbf{R}} + \mu B \frac{\Omega}{\theta} - H \right] - \int dV \frac{B_{\perp}^2}{8\pi}$$

where $dV = dW dV$ is the phase space element, with v-space and volume elements dW and dV

- generalised magnetic potential

$$\mathbf{A}^* = \mathbf{A} + A_{\parallel}^* \mathbf{b} \quad A_{\parallel}^* = A_{\parallel} + \frac{e}{mc} v_{\parallel}$$

with $\mathbf{B} = \nabla \times \mathbf{A} = B \mathbf{b}$ and $B_{\parallel}^* = \nabla \times \mathbf{A}^*$ and $B_{\parallel}^* = \mathbf{b} \cdot B_{\parallel}^*$

- generalised Hamiltonian

$$H = m \frac{v_{\parallel}^2}{2} + \mu B + e J_0 \phi - m \frac{v_{\perp}^2}{2}$$

- note v-space element includes Jacobian B_{\parallel}^*

$$\int dV = \int dV \int dW \quad \int dW = \int d v_{\parallel} d \mu B_{\parallel}^*$$

- gyroaveraging

$$J_0 = \frac{\Pi^4(1 - x^2/a_n^2)}{\Pi^3(1 + b_n x^2)}$$

approximates $J_0(x)$ to $x \approx 14$ with $x^2 = -\nabla \cdot \rho^T \nabla_{\perp}$

- ExB velocity and perturbed magnetic field

$$v_E^2 = \frac{c^2}{B^2} |\nabla_{\perp} \phi|^2 \quad B_{\perp}^2 = |\nabla_{\perp} A_{\parallel}|^2$$

- two important approximations

- A_{\parallel} is not gyroaveraged (belongs mostly to electrons)
- low- k_{\perp} form of screened potential (enters only at large scale)

FFI Equations

- gyrokinetic equation

$$\frac{\partial f}{\partial t} - \frac{e}{mc} \frac{\partial A_{\parallel}}{\partial v_{\parallel}} \frac{\partial f}{\partial v_{\parallel}} = [H, f] + C(f)$$

with general Poisson bracket

$$B_{*}^{\parallel}[H, f] = \frac{m}{B_{*}} \cdot \left(\frac{\partial f}{\partial v_{\parallel}} \Delta H - \frac{\partial H}{\partial v_{\parallel}} \Delta f \right) + \frac{eB}{c} \mathbf{F} : \nabla H \Delta f$$

with magnetic stress tensor $\mathbf{F} = \nabla \mathbf{A} - (\nabla \mathbf{A})^T = \epsilon \cdot \mathbf{B}$
and collision operator C added (see below)

- ExB polarisation, gyrokinetic Poisson equation

$$\sum^{\text{sp}} \int dW \left[e J_0 f + \Delta \cdot \frac{f m c^2}{B^2} \Delta^{\perp} \phi \right] = 0$$

- induction, gyrokinetic Ampere equation

$$0 = f \parallel v e \int dW \sum^{\text{sp}} \frac{e}{4\pi} + \Delta^{\perp} A_{\parallel}$$

FFFI Energy

- thermal, E_{XB} , and magnetic energy

$$\mathcal{E}_F = \int dV H_0 f \sum^{ds}$$

$$\mathcal{E}_E = \int dV f m \frac{v_E^2}{2} \sum^{ds} = \int dV e \phi f \sum^{ds}$$

$$\mathcal{E}_M = \int dV \frac{B_{\perp}^2}{8\pi} = \int dV \frac{2c}{e} A_{\parallel} v_{\parallel} f \sum^{ds}$$

where $H_0 = mv_{\parallel}^2/2 + \mu B$ and $e\phi = eJ_0\phi - mv_E^2/2$ using both field equations to evaluate \mathcal{E}_E and \mathcal{E}_M

$$0 = (f) \circ H \text{VP} \int \sum^{\text{ds}}$$

requiring also that

$$0 = [f, H] \text{VP} \int \sum^{\text{ds}} =$$

$$\left(\frac{\partial f}{\partial t} - \frac{mc}{e} \frac{\partial v_{\parallel}}{\partial t} \right) \text{VP} \int \sum^{\text{ds}} = \frac{\partial \mathcal{E}}{\partial t}$$

• energy conservation

$$\frac{\partial \mathcal{E}_M}{\partial t} \text{VP} \int \sum^{\text{ds}} = \frac{\partial \mathcal{E}}{\partial t}$$

and for \mathcal{E}_M use $\partial H / \partial v_{\parallel} = m v_{\parallel}$

$$\frac{\partial \mathcal{E}}{\partial t} \text{VP} \int \sum^{\text{ds}} = \frac{\partial \mathcal{E}_E}{\partial t}$$

$$\frac{\partial \mathcal{E}}{\partial t} \text{VP} \int \sum^{\text{ds}} = \frac{\partial \mathcal{E}_E}{\partial t}$$

• time dependence of energy pieces

FEFI Scheme for Induction

- how to handle $\partial A_{\parallel}/\partial t$, which must cancel $\nabla_{\parallel}\phi$ at large scale

- denote rest of equation as S

$$\frac{\partial f}{\partial t} - \frac{e}{mc} \frac{\partial f}{\partial v_{\parallel}} \frac{\partial A_{\parallel}}{\partial t} = S$$

- operate with integral in the induction equation

$$\left[-\Delta_{\perp}^2 \right] - \sum_{\text{sp}} \int dW \frac{4\pi e^2}{mc^2} v_{\parallel} \frac{\partial f}{\partial v_{\parallel}} \frac{\partial A_{\parallel}}{\partial t} = \frac{c}{4\pi} \sum_{\text{sp}} \int dW \frac{\partial f}{\partial v_{\parallel}} \frac{\partial A_{\parallel}}{\partial t} S$$

- solve this Helmholtz equation for time derivative, put on RHS

$$\frac{\partial f}{\partial t} = S + \frac{e}{mc} \frac{\partial f}{\partial v_{\parallel}} \frac{\partial A_{\parallel}}{\partial t}$$

hence finite mass corrections are responsible for the dynamics

$$0 = S \parallel v \int_{\mathbb{R}^3} d^3p \left(\frac{\partial f}{\partial t} \right) - S \parallel v \int_{\mathbb{R}^3} d^3p \left(\frac{\partial f}{\partial t} \right) = \frac{\partial f}{\partial t} \parallel v \int_{\mathbb{R}^3} d^3p$$

• for $S = (e/m)(\phi \parallel \Delta)(\partial f / \partial v \parallel)$ the $v \parallel$ -moment of this cancels to machine accuracy

$$S \parallel v \int_{\mathbb{R}^3} d^3p \left(\frac{\partial f}{\partial t} \right) - S = \frac{\partial f}{\partial t} \parallel v \int_{\mathbb{R}^3} d^3p$$

• evaluate for electrons only, at large scale ($\Delta_{\perp}^2 \rightarrow 0$)

$$\mathcal{D} = \left[-\Delta_{\perp}^2 \right] \int_{\mathbb{R}^3} d^3p \left(\frac{\partial f}{\partial t} \right)$$

where \mathcal{D} is the operator in $[\dots]$ above

$$\frac{\partial f}{\partial t} \parallel v \int_{\mathbb{R}^3} d^3p + S = \frac{\partial f}{\partial t} \parallel v \int_{\mathbb{R}^3} d^3p \mathcal{D}$$

• this RHS is actually an integro-differential of S

FEFI Geometry

- global version of flux tube form with shifted metric

- Hamada coordinates V, θ, ζ with the $B_{\mu} =$ functions of V only

- field aligning transform (minor radius a for normalisation)

$$x = V/a^3 \quad y = q\theta - \zeta \quad s = \theta$$

with pitch parameter $q = B_{\zeta}/B^{\theta}$

- set such that only B^s is nonzero

- coordinate range, finite chunk of torus, poloidal connection

$$x_0 > x > x_9 \quad - \quad \frac{1}{2}k_0 > y > \frac{1}{2}k_0 \quad - \quad \frac{1}{2} > s > \frac{1}{2}$$

with radial boundaries x_0, x_9 and toroidal truncation integer k_0

- shifted metric transform

$$y_k = q(\theta) - s_k - \zeta - \Delta\alpha_k$$

with s_k global constant and $\Delta\alpha_k$ a function of V only

- field aligning: only one B_μ is nonzero (B_s)

- perp dynamics on orthogonal grid: $g_{xy}^k = 0$ at each evaluation

- evaluate from simple model or Grad-Shafranov solution

◦ model profiles for pressure and current using HELENA code

- necessary quantities (functions of x and s)

$$g_{xx} \quad g_{yy}^k \quad \Delta\alpha_k \quad B_s \quad B$$

giving also (i.a.) the components of magnetic stress tensor \mathbf{F}

- reference: B Scott, *Phys Plasmas* **8** (2001) 447

Collision Operator for FFI

- like particles: pitch angle and v_{\parallel} scattering

$$C_{ij} = \frac{\partial}{\partial v_{\parallel}} G_{\parallel}(v_{\parallel}) \left[(v_{\parallel} - U) + W \frac{\partial}{\partial v_{\parallel}} \right] + G(v) \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial}{\partial \zeta}$$

with U, W set to conserve energy, momentum

- G_{\parallel} and G are model potentials (including v_j and v_{-j} dependence)

- $\zeta = v_{\parallel}/v$ is the pitch angle

- v_{\parallel} and ζ are reckoned in c/m frame

- e-i: simple pitch angle scattering off the background

- damps momentum to ions, conserves energy

- add a hypercollisionality v_z to $C(f)$ for collisionless problems

$$C(f) = \dots - v_z \frac{\partial}{\partial f}$$

where $z = v_{\parallel}/V$ is the normalised coordinate (see below)

- normalised velocities used in FEF1

$$z = \frac{V}{v_{\parallel}} \quad w = \frac{v_{\perp}^2}{2V^2} = \frac{\mu_B}{V^2}$$

where V is a constant, noting v-space metric independent of B

- relation of variables (v, ζ) to (z, w)

$$z = v\zeta \quad w = \frac{v^2}{2} (1 - \zeta^2)$$

where we also normalise v to V

- pitch angle derivative, note z commutes with $\partial/\partial w$

$$\frac{\partial \zeta}{\partial v} = \frac{\partial \zeta}{\partial z} \frac{\partial z}{\partial v} + \frac{\partial \zeta}{\partial w} \frac{\partial w}{\partial v} = \left(\frac{\partial \zeta}{\partial z} - \frac{\partial \zeta}{\partial w} \frac{z}{v} \right) v$$

- do G_{\parallel} and G pieces separately, also e-1 for electrons
- set up in conservative finite volume form

Method for Conservation in Collisions

- evaluate v-space fluxes in cell centers, in 1-D in norm units

$$C_i = \frac{\partial}{\partial v_{\parallel}} G(v_{\parallel}) \left[(v_{\parallel} - U) + W \frac{\partial}{\partial v_{\parallel}} \right] \Big|_{z=z_i} = \frac{1}{\Gamma_i} \frac{\Delta z}{\Gamma_i - \Gamma_{i-1}}$$

where $dz = z_i - z_{i-1}$

- momentum conservation

$$0 = \sum_i C_i z_i = \sum_i C_i \frac{\Delta z}{z_i - z_{i+1}} = (\Gamma_i - \Gamma_{i-1}) \sum_i \frac{\Delta z}{z_i} = \sum_i C_i z_i = \sum_i \Gamma_i z_i = 0$$

shifting index on Γ_{i-1} term

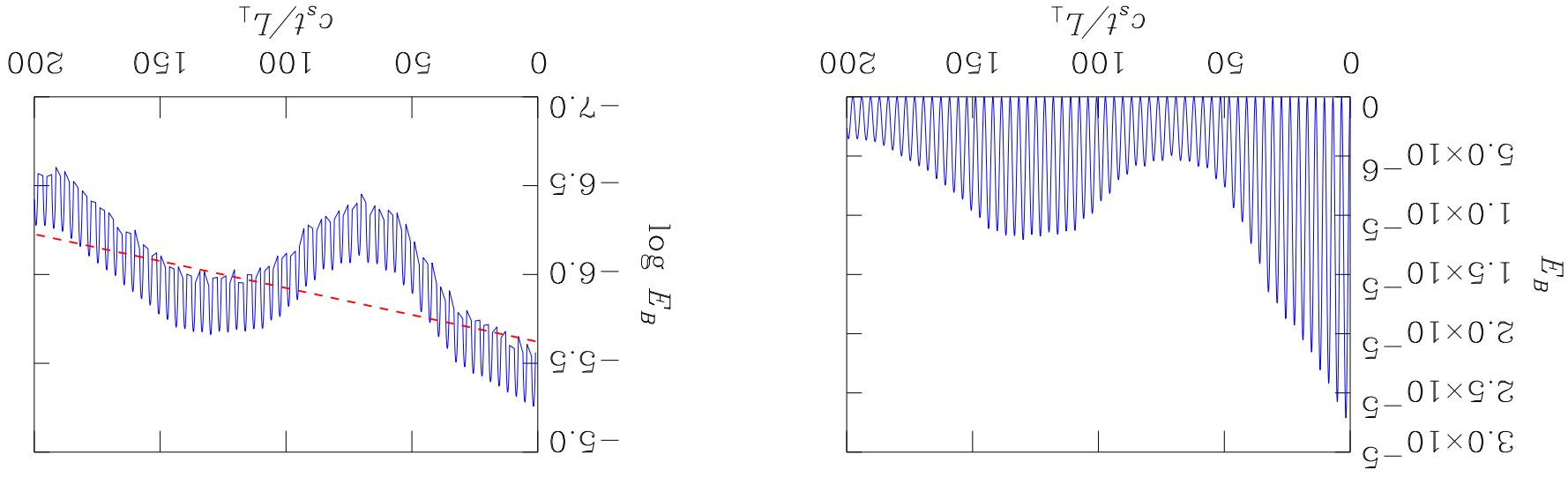
- energy conservation, uses z_i^2 , note $z_{i+1}^2 - z_i^2 = \Delta z (z_{i+1} + z_i)$

$$0 = \sum_i \Gamma_i z_i^2 = \sum_i C_i \frac{z_i^2}{2}$$

- two conditions for two unknowns U and W , coupled through G_{\parallel}
 - coefficients are various moments over combinations of f and G_{\parallel}
- pitch angle scattering set up to conserve momentum in 2-D
 - energy conservation follows automatically
- e- \perp pitch angle scattering damps current naturally
 - energy conservation follows if mean velocity for the background is zero

Kinetic Shear Alfvén Damping, Recursion

FEF13, $\beta_e = 10^{-4}$, $\mu_e^{-1} = 3670$, $qR/L_\perp = 100$, $k_\perp \rho_s = 0.1$

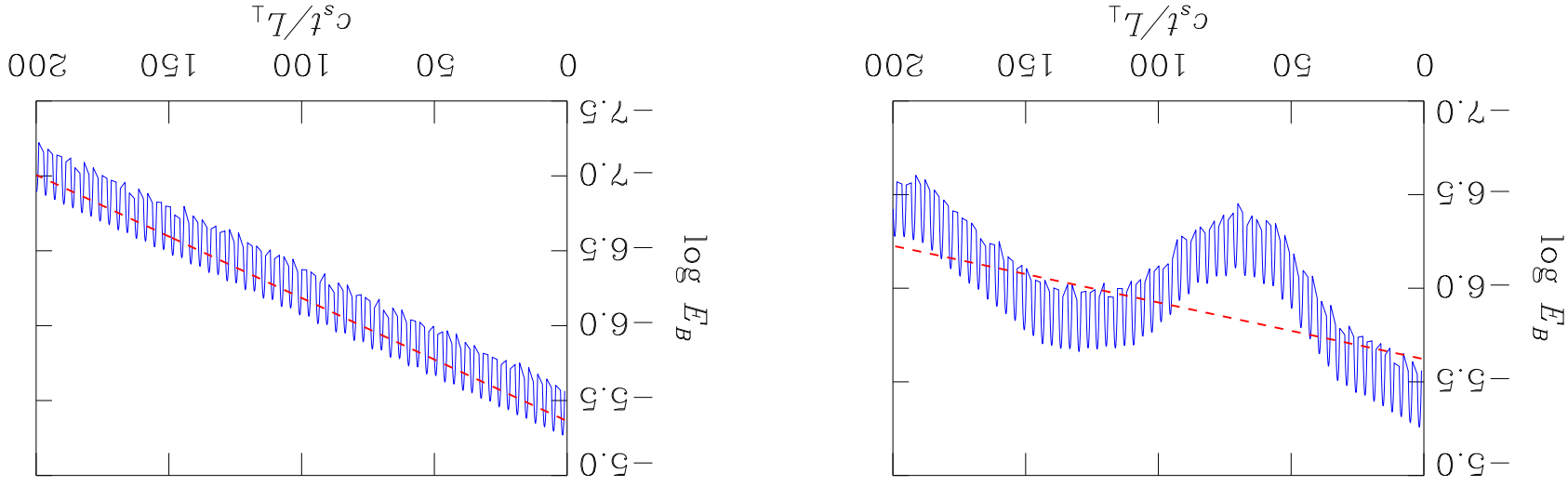


- recursion after $t = 40$, best seen in decay curve (right)
- non constant damping rate
- caused by velocity-space resolution problems (32 nodes in v_\parallel , with $v_z = 10^{-5}$)

Kinetic Shear Alfvén Damping, Physical

FEF13, $\beta_e = 10^{-4}$ $\mu_e^{-1} = 3670$ $qR/L_\perp = 100$ $k_\perp \rho_s = 0.1$

- increase v_\parallel -resolution from 32 (left) to 64 (right) per $10V_e$

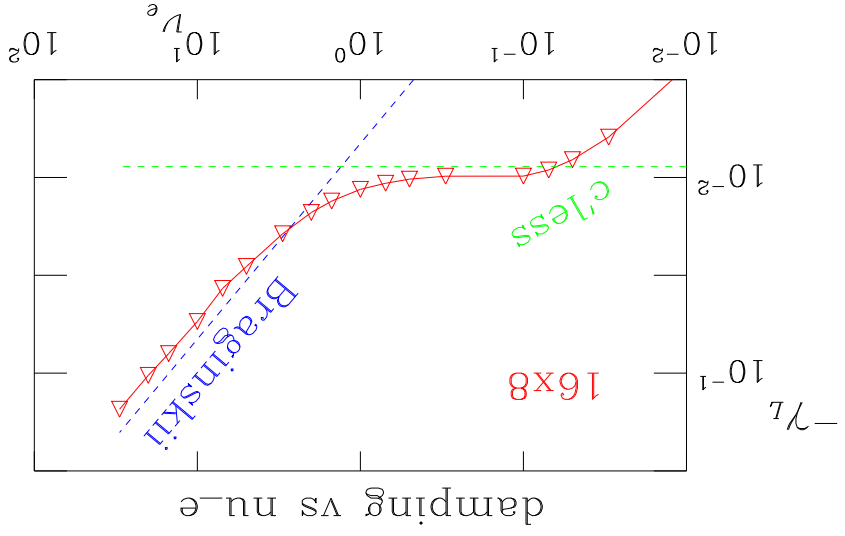
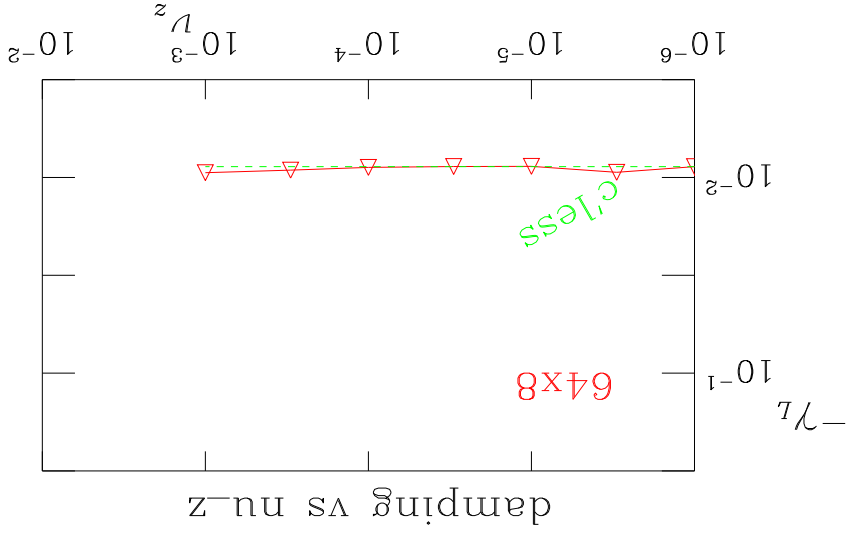


- well formed decay curve even at late times
- constant damping rate, good fit to part of curve near peaks
- velocity-space resolution sufficient (64 nodes in v_\parallel , with $v_z = 10^{-5}$)

KALF Damping Rate, collisionality scaling

FEF13, $\beta_e = 10^{-4}$, $\mu_e^{-1} = 3670$, $qR/L_\perp = 100$, $k_\perp \rho_s = 0.1$

- collisionless (left) and trans-collisional (right)



- recursion problems for $\nu^z > 10^{-5}$
- non-thermalisation for $\nu^z > 10^{-3}$
- well converged for $N^z = 64$

- collisional regime for $\nu_e > 1$
- ν_\parallel -space grid problems for $\nu_e > 0.1$
- usual range for edge turbulence is $\nu_e \sim 1$

Main Points

- **Gyrokinetic Equation**
 - Hamiltonian structure follows Lagrangian formulation
 - polarisation equation follows from field variation
- **Energy Conservation**
 - split between particle and field Hamiltonian
 - polarisation equation \rightarrow ExB energy
 - magnetic potential \rightarrow magnetic energy
 - simple bracket structure
 - depends on phase space conservation
- **FFI Code**
 - large scale Alfvén problems solved
 - combination collisions/Alfvén/trapping conserves energy well
 - currently adding nonlinear brackets, expect to finish in 2004
- **Main Targets**
 - nonlinear/nonlocal pedestal physics ($100 \text{ eV} < T_e < 1 \text{ keV}$)
 - edge and core zonal flow energetics
 - neoclassical tokamak equilibrium, perhaps also stellarators