

Turbulence in Magnetised Plasmas

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Turbulence in Magnetised Plasmas

- nonlinearity of small disturbances on an equilibrium
 - † three wave interactions
 - † energy transfer, cascading
- incompressible turbulence models
 - † simple fluid turbulence, role of pressure to maintain incompressibility
 - † cascades of energy, vorticity (“enstrophy”), role of vortex tubes in 2D and 3D models
 - † 2D MHD turbulence, role of magnetic field to maintain incompressibility
- simple drift effects in a magnetised plasma with gradients
 - † dissipative coupling, effect on cascades
 - † evolution of spectra, physical meaning of cascades
 - † varying properties of nonlinear couplings
- the transport problem

Various Nonlinear Effects

- rapid space/time variation of parameters (e.g., shocks, isolated jets)
- quasilinear interaction between small waves to alter the background
 - † each wave (\mathbf{k}) beats against itself (\mathbf{k}')
 - † background is wavenumber zero
$$\mathbf{k} + \mathbf{k}' = 0$$
- turbulence — incoherent interaction with many wave combinations
 - † each wave (\mathbf{k}) is forced upon by two other beat waves (\mathbf{k}' and \mathbf{k}'')
 - † many distinct pairs $\{\mathbf{k}', \mathbf{k}''\}$ with no relation to \mathbf{k}
$$\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = 0$$

many degrees of freedom, incoherent, statistical

Small Disturbances on an Equilibrium

- ordering in general — gradients multiplied by constant parameters

$$\Delta_{\perp} \ll L_{\perp} \quad \implies \quad (p_e + \tilde{p}_e) \nabla \cdot \mathbf{v} \rightarrow p_e \nabla \cdot \mathbf{v}$$

- background may be inhomogeneous (define x as down-gradient)

$$\nabla p_e \rightarrow -\frac{p_e}{L_{\perp}} \nabla x \quad \text{where} \quad \nabla x = -L_{\perp} \nabla \log p_e$$

- “mixing level” disturbances

$$\nabla \tilde{p}_e \sim \nabla p_e \implies \frac{\tilde{p}_e}{p_e} \sim \frac{\Delta_{\perp}}{L_{\perp}} \ll 1$$

- nonlinearity remains in advection effects — a nonlinear term and a linear forcing term

$$\mathbf{v}_E \cdot \nabla (p_e + \tilde{p}_e) = \frac{c}{B} \mathbf{b} \cdot \nabla \tilde{\phi} \times \nabla \tilde{p}_e - \frac{p_e}{L_{\perp}} v_E^x \quad \text{where} \quad v_E^x = \frac{c}{B} \mathbf{b} \cdot \nabla \tilde{\phi} \times \nabla x$$

keep nonlinearities where quadratic under gradients

Incompressible Hydrodynamics

- start with MHD, neglect magnetic field

$$\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\frac{1}{\rho} \nabla p$$

- take curl, treat ρ as constant, neglect $\nabla \cdot \mathbf{v}$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \nabla \times \mathbf{v} = (\nabla \times \mathbf{v}) \cdot \nabla \mathbf{v}$$

- pressure submerges — only role is to maintain incompressibility

$$\text{let } \frac{\partial}{\partial t} \nabla \cdot \mathbf{v} = 0 \quad \text{then } \nabla^2 p = -\nabla \cdot (\rho \mathbf{v} \cdot \nabla \mathbf{v})$$

- leads to “projection methods” for computations

Enstrophy in Incompressible Hydrodynamics

- Euler equation in 3D

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \nabla \times \mathbf{v} = (\nabla \times \mathbf{v}) \cdot \nabla \mathbf{v}$$

- note mean squared vorticity (“enstrophy”) is not generally conserved

$$\frac{\partial W}{\partial t} + \nabla \cdot (W \mathbf{v}) = [(\nabla \times \mathbf{v})(\nabla \times \mathbf{v})] : [\nabla \mathbf{v}] \quad \text{where} \quad W = \frac{1}{2} (\nabla \times \mathbf{v}) \cdot (\nabla \times \mathbf{v})$$

- enstrophy is transported by the velocity, but grows if ...

† the velocity has a component along the vorticity, and also diverges in that direction

vortex tube stretching in 3D

2D Incompressible Hydrodynamics

- in 2D one must have $\nabla \times \mathbf{v} \perp \mathbf{v} \dots$ let $\hat{\mathbf{s}}$ be the normal to the plane

$$\nabla \cdot \mathbf{v} = 0 \quad \implies \quad \mathbf{v} = \hat{\mathbf{s}} \times \nabla \psi \quad \implies \quad (\nabla \times \mathbf{v}) = \hat{\mathbf{s}} \nabla_{\perp}^2 \psi$$

- find the 2D Euler equation

$$\frac{\partial \Omega}{\partial t} + \mathbf{v} \cdot \nabla \Omega = 0 \quad \text{with} \quad \Omega = \nabla_{\perp}^2 \psi \quad \text{and} \quad \mathbf{v} = \hat{\mathbf{s}} \times \nabla \psi$$

- hence the enstrophy (W) is conserved, along with the energy (U)

$$\text{let } W = \frac{\Omega^2}{2} \quad \text{then} \quad \frac{\partial W}{\partial t} + \nabla \cdot (W \mathbf{v}) = 0$$

$$\text{let } U = \frac{v^2}{2} \quad \text{then} \quad \frac{\partial U}{\partial t} + \nabla \cdot (U \mathbf{v}) = 0$$

both are conserved with same flow field

The Three Wave Interaction

- start with the 2D Euler equation

$$\frac{\partial \Omega}{\partial t} + \mathbf{v} \cdot \nabla \Omega = 0$$

- define Fourier decomposition

$$\psi = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \psi_{\mathbf{k}} \qquad \psi_{\mathbf{k}} = \oint \frac{k^2 d^2 x}{4\pi^2} e^{-i\mathbf{k} \cdot \mathbf{x}} \psi \qquad \psi_{(-\mathbf{k})} = \psi_{\mathbf{k}}^*$$

- Euler equation in \mathbf{k} -space

$$\frac{\partial \Omega_{\mathbf{k}}}{\partial t} = \hat{\mathbf{s}} \cdot \oint \frac{k^2 d^2 x}{4\pi^2} e^{-i\mathbf{k} \cdot \mathbf{x}} \sum_{-\mathbf{k}'} \sum_{-\mathbf{k}''} e^{-i\mathbf{k}' \cdot \mathbf{x}} e^{-i\mathbf{k}'' \cdot \mathbf{x}} (i\mathbf{k}') \times (i\mathbf{k}'') \Omega_{-\mathbf{k}'} \psi_{-\mathbf{k}''}$$

- three wave condition for the integral not to vanish

$$\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = 0$$

Equations for Beat Waves

- Euler equation

$$\frac{\partial \Omega_{\mathbf{k}}}{\partial t} = \sum_{-\mathbf{k}'} \sum_{-\mathbf{k}''} \frac{1}{2} \hat{\mathbf{s}} \cdot (\mathbf{k} \times \mathbf{k}') (\Omega_{-\mathbf{k}''} \psi_{-\mathbf{k}'} - \Omega_{-\mathbf{k}'} \psi_{-\mathbf{k}''})$$

- for beat waves use symmetry

$$\hat{\mathbf{s}} \cdot (\mathbf{k} \times \mathbf{k}') = \hat{\mathbf{s}} \cdot (\mathbf{k}' \times \mathbf{k}'') = \hat{\mathbf{s}} \cdot (\mathbf{k}'' \times \mathbf{k})$$

- define coupling matrix

$$C_{\mathbf{k}\mathbf{k}'} = \frac{1}{2} \hat{\mathbf{s}} \cdot (\mathbf{k} \times \mathbf{k}')$$

- find beat wave equations (use permutation among $\mathbf{k}, \mathbf{k}', \mathbf{k}''$ triangle)

$$\frac{\partial \Omega_{\mathbf{k}}}{\partial t} = C_{\mathbf{k}\mathbf{k}'} (\Omega_{-\mathbf{k}''} \psi_{-\mathbf{k}'} - \Omega_{\mathbf{k}'} \psi_{-\mathbf{k}'})$$

$$\frac{\partial \Omega_{\mathbf{k}'}}{\partial t} = C_{\mathbf{k}\mathbf{k}'} (\Omega_{-\mathbf{k}} \psi_{-\mathbf{k}''} - \Omega_{\mathbf{k}''} \psi_{-\mathbf{k}})$$

$$\frac{\partial \Omega_{\mathbf{k}''}}{\partial t} = C_{\mathbf{k}\mathbf{k}'} (\Omega_{-\mathbf{k}'} \psi_{-\mathbf{k}} - \Omega_{\mathbf{k}} \psi_{-\mathbf{k}'})$$

Energy Transfer

- find energy transfer by multiplying by $-\psi_{\mathbf{k}}$ and adding complex conjugate

$$\frac{\partial U_{\mathbf{k}}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [\psi_{\mathbf{k}} \Omega_{\mathbf{k}'} \psi_{\mathbf{k}''} - \psi_{\mathbf{k}} \psi_{\mathbf{k}'} \Omega_{\mathbf{k}''}]$$

$$\frac{\partial U_{\mathbf{k}'}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [\psi_{\mathbf{k}'} \Omega_{\mathbf{k}''} \psi_{\mathbf{k}} - \psi_{\mathbf{k}'} \psi_{\mathbf{k}''} \Omega_{\mathbf{k}}]$$

$$\frac{\partial U_{\mathbf{k}''}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [\psi_{\mathbf{k}''} \Omega_{\mathbf{k}} \psi_{\mathbf{k}'} - \psi_{\mathbf{k}''} \psi_{\mathbf{k}} \Omega_{\mathbf{k}'}]$$

- identify transfer channel as terms with opposite sign in one pair of equations, e.g.,

$$T_U(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [-\psi_{\mathbf{k}} \psi_{\mathbf{k}'} \Omega_{\mathbf{k}''}]$$

Enstrophy Transfer

- find enstrophy transfer by multiplying by $\Omega_{\mathbf{k}}$ and adding complex conjugate

$$\frac{\partial W_{\mathbf{k}}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [\Omega_{\mathbf{k}} \psi_{\mathbf{k}'} \Omega_{\mathbf{k}''} - \Omega_{\mathbf{k}} \Omega_{\mathbf{k}'} \psi_{\mathbf{k}''}]$$

$$\frac{\partial W_{\mathbf{k}'}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [\Omega_{\mathbf{k}'} \psi_{\mathbf{k}''} \Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'} \Omega_{\mathbf{k}''} \psi_{\mathbf{k}}]$$

$$\frac{\partial W_{\mathbf{k}''}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [\Omega_{\mathbf{k}''} \psi_{\mathbf{k}} \Omega_{\mathbf{k}'} - \Omega_{\mathbf{k}''} \Omega_{\mathbf{k}'} \psi_{\mathbf{k}}]$$

- identify transfer channel as terms with opposite sign in one pair of equations, e.g.,

$$T_W(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [-\Omega_{\mathbf{k}} \Omega_{\mathbf{k}'} \psi_{\mathbf{k}''}]$$

The Dual Cascade

- write energy and enstrophy transfer

$$T_U(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [-\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\Omega_{\mathbf{k}''}] = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [(k'')^2\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\psi_{\mathbf{k}''}]$$

$$T_W(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [-\Omega_{\mathbf{k}}\Omega_{\mathbf{k}'}\psi_{\mathbf{k}''}] = 2C_{\mathbf{k}\mathbf{k}'} \operatorname{Re} [-k^2(k')^2\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\psi_{\mathbf{k}''}]$$

- note that given a definite sign, they are opposite!
- statistically, enstrophy goes to higher k , hence smaller scale, due to the larger k -dependence
 - † faster mixing, spectral redistribution
- hence energy goes preferentially to lower k , hence *larger scale*

2D inverse energy cascade

- “maximum entropy” stationary states for discrete systems show $W_k \sim k$ and $U_k \sim k^{-1}$

A Passive Scalar

- density fluctuations follow incompressible equation

$$\frac{\partial \tilde{\rho}}{\partial t} + \mathbf{v} \cdot \nabla \tilde{\rho} = 0$$

- passive scalar: $\tilde{\rho}$ is advected by the flow, but effects no back reaction
- in \mathbf{k} -space the density equation is the same as for the vorticity
- “fluctuation free energy” or “entropy” is defined by squared amplitude
- hence the free energy transfer has the same form as for enstrophy

Flow energy to large scales, free energy to small

- very high correlation $\tilde{\Omega} \leftrightarrow \tilde{\rho}$ in forced/dissipative turbulence, even with no coupling effects

Incompressible MHD

- constant parameters, homogeneous background, keep only quadratic nonlinearities

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}$$

- set $\mathbf{B} = B\mathbf{b}$ and $\mathbf{u} = \mathbf{v}/v_A$ with $v_A^2 = B^2/4\pi\rho$

$$\frac{1}{v_A} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{B^2} \nabla \left(4\pi p + \frac{B^2}{2} \right) + \mathbf{b} \cdot \nabla \mathbf{b}$$

- incompressible MHD kinematic equation

$$\frac{1}{v_A} \frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u}$$

- define “Elsässer variables” $\mathbf{u}_\pm = \mathbf{u} \pm \mathbf{b}$
- find passive advection, but coupled through advector (note $\nabla \cdot \mathbf{v} = 0 \leftrightarrow B^2$ not p , for $\beta \ll 1$)

$$\frac{1}{v_A} \frac{\partial \mathbf{u}_\pm}{\partial t} + \mathbf{u}_\mp \cdot \nabla \mathbf{u}_\pm = -\frac{1}{B^2} \nabla \left(4\pi p + \frac{B^2}{2} \right)$$

2D Incompressible MHD

- constant parameters, homogeneous background, keep only quadratic nonlinearities

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}$$

- take curl, use 2D to avoid $(\nabla \times \mathbf{v}) \cdot \mathbf{v}$ and $\mathbf{J} \cdot \nabla \mathbf{B}$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \nabla \times \mathbf{v} = \frac{1}{c} \mathbf{B} \cdot \nabla \mathbf{J}$$

- define $\mathbf{E} \times \mathbf{B}$ velocity and vorticity, parallel current, parallel gradient

$$\mathbf{v} = \mathbf{v}_E = \frac{c}{B^2} \mathbf{B} \times \nabla \phi \qquad \Omega = \frac{\rho c^2}{B^2} \nabla_{\perp}^2 \phi \qquad J_{\parallel} = \mathbf{b} \cdot \mathbf{J}$$

- find correction to Euler vorticity equation

$$\frac{\partial \Omega}{\partial t} + \mathbf{v}_E \cdot \nabla \Omega = \mathbf{b} \cdot \nabla J_{\parallel}$$

applications of 2D incompressible MHD

- usually formulated with Elsässer variables: $\mathbf{u}_{\pm} = \mathbf{u} \pm \mathbf{b}$
- define velocity and magnetic field

$$\mathbf{u} = \hat{\mathbf{s}} \times \nabla \phi \quad \mathbf{b} = -\hat{\mathbf{s}} \times \nabla \psi$$

- resistive (η), viscous (μ) MHD equations in Alfvén normalisation ($\partial/\partial t \leftrightarrow v_A \nabla$)

$$\frac{\partial \mathbf{u}_{\pm}}{\partial t} + \mathbf{u}_{\mp} \cdot \nabla \mathbf{u}_{\pm} = -\nabla I + (\mu \pm \eta) \nabla_{\perp}^2 \mathbf{u}_{\pm}$$

- incompressibility potential

$$\nabla^2 I + \nabla \cdot (\mathbf{u}_{\mp} \cdot \nabla \mathbf{u}_{\pm}) = 0$$

- this dynamical system is commonly used in astrophysics (e.g., reconnection, dynamo)
- for turbulence within an MHD stable equilibrium, the drive source is ∇p

† coupling processes specifically in the electrons $p_e \leftrightarrow \phi$ become significant

† and the MHD model cannot cover the physics ...

Dissipative Coupling

- beyond MHD, density is not passive, but coupled through parallel currents to the ExB vorticity
- Ohm's law, parallel, keeping electron pressure gradient

$$-E_{\parallel} = \nabla_{\parallel} \tilde{\phi} = \frac{1}{n_e e} \nabla_{\parallel} \tilde{p}_e - \eta_{\parallel} \tilde{J}_{\parallel}$$

- parallel compressibility enters electron pressure equation (advection is by the ExB velocity)

$$\frac{\partial \tilde{p}_e}{\partial t} + \mathbf{v}_E \cdot \nabla (p_e + \tilde{p}_e) = \frac{T_e}{e} \nabla_{\parallel} \tilde{J}_{\parallel}$$

- appears as parallel diffusivity but couples to $\tilde{\phi}$

$$\frac{\partial \tilde{p}_e}{\partial t} + \mathbf{v}_E \cdot \nabla (p_e + \tilde{p}_e) = \frac{T_e}{n_e e^2 \eta_{\parallel}} \nabla_{\parallel}^2 (\tilde{p}_e - n_e e \tilde{\phi})$$

- note that $\nabla_{\parallel} \tilde{p}_e \sim n_e e \nabla_{\parallel} \tilde{\phi}$ is the *usual situation* in gradient driven turbulence

† it cannot be treated by the single fluid MHD model

Dissipative Coupling Model for ExB Turbulence

- electrostatic approximation for $\omega \ll k_{\perp} v_A$

$$\mathbf{E}_{\perp} = -\nabla_{\perp} \phi$$

- electrostatic potential is stream function for ExB velocity

$$\mathbf{v}_E = \frac{c}{B^2} \mathbf{B} \times \nabla \phi \quad \nabla \times \frac{\rho c}{B} \mathbf{v}_E = \Omega \mathbf{b}$$

- vorticity equation is the same as for MHD, with parallel gradient reckoned against the background

$$\frac{\partial \Omega}{\partial t} + \mathbf{v}_E \cdot \nabla \Omega = \nabla_{\parallel} J_{\parallel}$$

- changes are in the dissipative Ohm's law ... and in the electron pressure equation

$$\eta_{\parallel} J_{\parallel} = \frac{1}{n_e e} \nabla_{\parallel} p_e - \nabla_{\parallel} \phi \quad \frac{\partial p_e}{\partial t} + \mathbf{v}_E \cdot \nabla p_e = \frac{T_e}{e} \nabla_{\parallel} J_{\parallel}$$

- with J_{\parallel} as a function of p_e and ϕ , the system is closed

Dissipative Coupling Model, properly 2D

- with no magnetic fluctuations, ∇_{\parallel} is slightly cheating
- actual dynamics is 3D, perp incompressible, J_{\parallel} dynamics along \mathbf{B} to provide coupling
- answer: model $-\nabla_{\parallel}^2$ with a positive coupling constant, with units of frequency

$$D = \frac{T_e}{n_e e^2 \eta_{\parallel}} k_{\parallel}^2 \quad \text{which with } \eta_{\parallel} = 0.51 \frac{m_e \nu_e}{n_e e^2} \quad \text{becomes } D = \frac{V_e^2}{0.51 \nu_e} k_{\parallel}^2$$

where $V_e = \sqrt{T_e/m_e}$ is the electron thermal velocity

- scale fluctuations as $e\tilde{\phi}/T_e$ and \tilde{p}_e/p_e , use $\rho = n_i M_i$ and $n_i = n_e$
- resulting model is called “Hasegawa-Wakatani”

$$\frac{e^2 M_i T_e}{e^2 B^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \nabla_{\perp}^2 \frac{e\tilde{\phi}}{T_e} = D \left(\frac{\tilde{p}_e}{p_e} - \frac{e\tilde{\phi}}{T_e} \right)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \frac{\tilde{p}_e}{p_e} + \mathbf{v}_E \cdot \nabla \log p_e = D \left(\frac{\tilde{p}_e}{p_e} - \frac{e\tilde{\phi}}{T_e} \right)$$

Dissipative Coupling Model, notes

- we've used a static, resistive, current
 - † neglects magnetic induction \leftrightarrow effects of $\partial\mathbf{B}/\partial t$, fails if $\omega \sim k_{\parallel}v_A$
- we've still used the ExB velocity for both ions and electrons, perp to \mathbf{B}
 - † for MHD the only restriction is that electrostatic form requires $\omega \ll k_{\perp}v_A$
 - † in general we require cold ions to use the ExB inertia term
$$n_e e \nabla_{\perp} \phi \sim \nabla_{\perp} p_e \ll \nabla_{\perp} p_i \quad \text{requires } T_i \ll T_e$$
- we've assumed isothermal electrons in the \tilde{p}_e equation
 - † constant mass density is still OK if $\tilde{p}_e \ll p_e$
 - † generally, \tilde{T}_e is required but adds no qualitative changes, hence neglected in simplest model
- use of cold ions allows neglect of finite gyroradius effects and still reach down to drift scale
- we've neglected sound wave effects, reasonable if $k_{\parallel}L_{\perp} \ll 1$
- note that to *compare NUMBERS to an experiment* requires absolute complexity

Scales in the Dissipative Coupling Model

- the Hasegawa-Wakatani equations: dissipative coupling and gradient forcing

$$\frac{c^2 M_i T_e}{e^2 B^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \nabla_{\perp}^2 \frac{e\tilde{\phi}}{T_e} = D \left(\frac{\tilde{p}_e}{p_e} - \frac{e\tilde{\phi}}{T_e} \right)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \frac{\tilde{p}_e}{p_e} + \mathbf{v}_E \cdot \nabla \log p_e = D \left(\frac{\tilde{p}_e}{p_e} - \frac{e\tilde{\phi}}{T_e} \right)$$

- introduces the drift scale ρ_s , defined by

$$\rho_s^2 = c^2 M_i T_e / e^2 B^2$$

- gradient forcing gives the time scale L_{\perp}/c_s , from the sound speed c_s and profile scale length L_{\perp}

$$c_s^2 = \frac{T_e}{M_i} \qquad L_{\perp} = |\nabla \log p_e|^{-1}$$

- most interesting effects come from the varying properties of the two nonlinearities ...

Computational Dissipative Coupling Model

- normalise in terms of ρ_s and c_s/L_\perp , scale variables by a factor of $\delta = \rho_s/L_\perp$

$$\phi \leftarrow \delta^{-1} e\tilde{\phi}/T_e \quad p \leftarrow \delta^{-1} \tilde{p}_e/p_e \quad \Omega \leftarrow \delta^{-1} \rho_s^2 \nabla_\perp^2 \left(e\tilde{\phi}/T_e \right)$$

- only parameter is $D \leftarrow DL_\perp/c_s$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \Omega = D (p - \phi)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) p = -\frac{\partial \phi}{\partial y} + D (p - \phi)$$

- ExB advection defined in terms of a Poisson bracket structure, e.g.,

$$\mathbf{v}_E \cdot \nabla p = [\phi, p] = \frac{\partial \phi}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial p}{\partial x}$$

- linear forcing terms are the dissipative coupling (D) and the gradient drive: $v_E^x = -\partial \phi / \partial y$

Illustration of Dual Cascade

- periodic domain, $(20\pi \rho_s)^2$
- examine decaying turbulence started in middle of spectrum (set gradient drive to zero)

$$p_{\mathbf{k}}(0) = \phi_{\mathbf{k}}(0) = a_0 [1 + (k_{\perp}^2/0.32)^4]^{-1/2} e^{i\Theta}$$

† random phase Θ

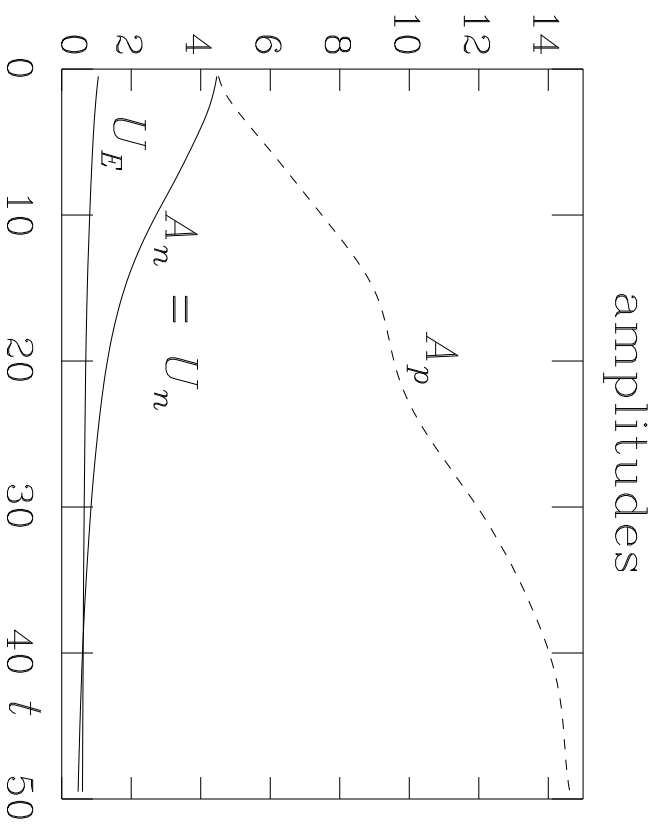
† a_0 chosen such that rms amplitude is 3.0

- test “hydrodynamic” limit $D = 0$

† Euler equation for Ω , passive advection for p

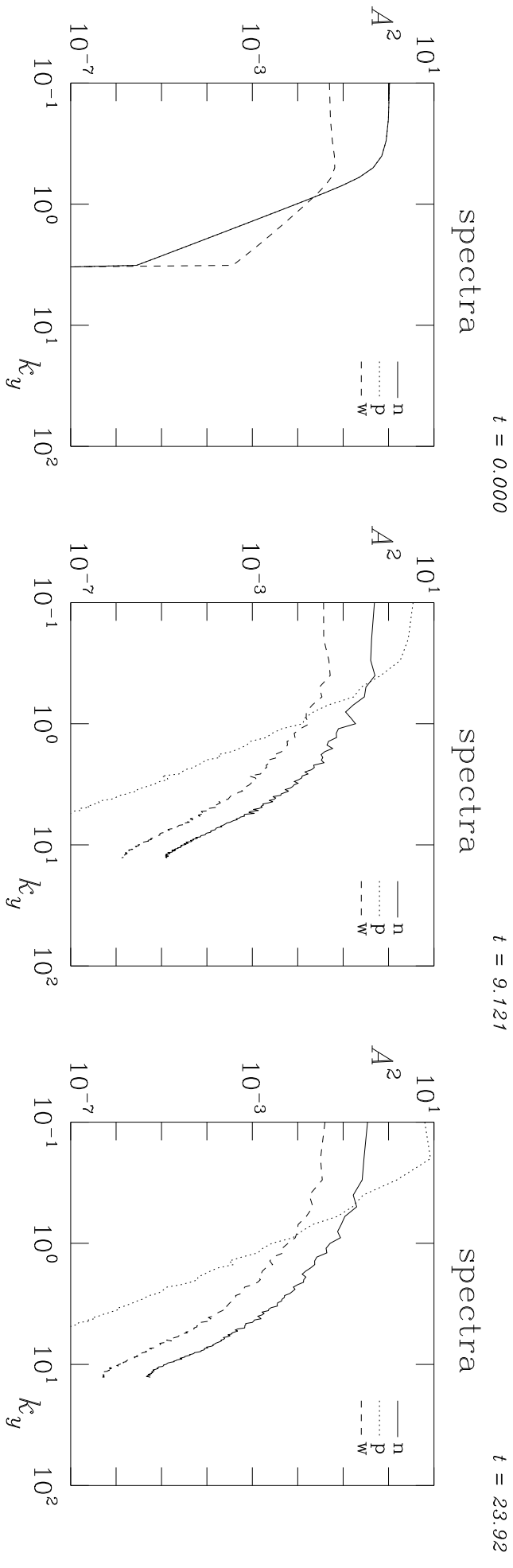
- note in some of the figures label for p is n_e

- Time evolution of the hydrodynamic model



- initial decay of half squared amplitudes of p and ϕ , denoted A_n and A_p , respectively
 - † also ExB energy (U_E) and fluctuation free energy (U_n)
- energetic losses (mostly in p due to the direct cascade) for three values of the resolution

- Amplitude spectra in the hydrodynamic model, for p , ϕ , and Ω ('n', 'p', and 'w')

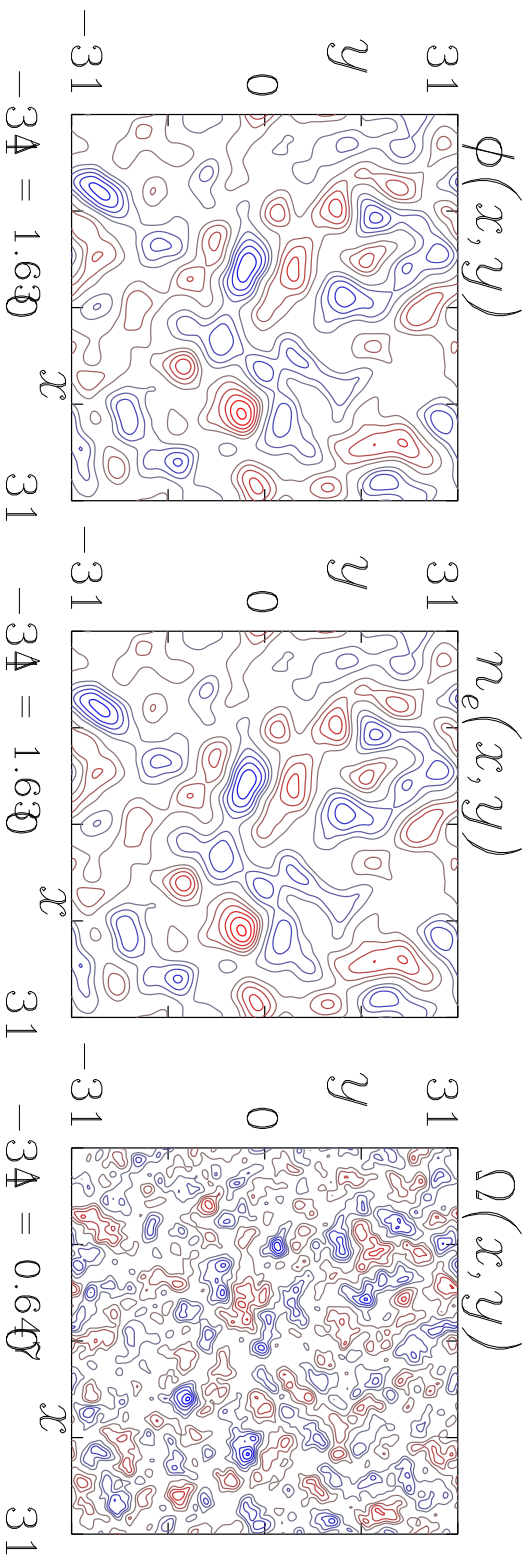


- times of the snapshots are $t = 0$ (left), $t = 9.8$ (center), and $t = 24$ (right)
- the spectra evolve rapidly apart due to the differing cascade dynamics for p and Ω versus ϕ

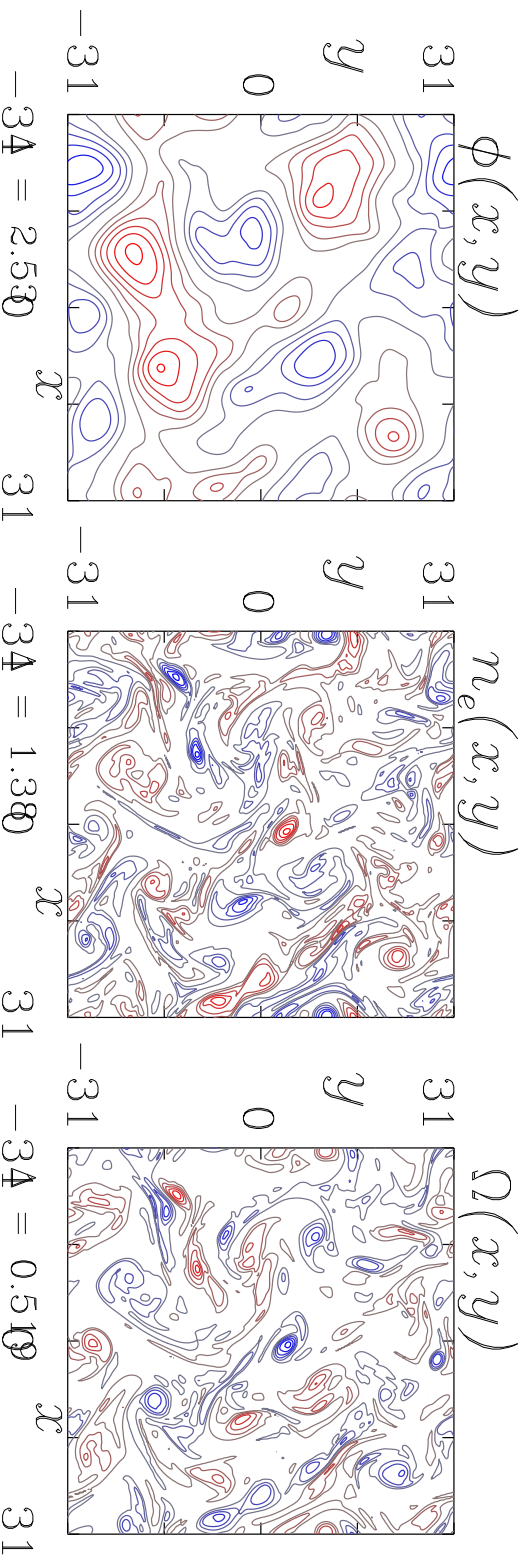
demonstration of bi-directional spectral transfer

- Evolution of the disturbances for the hydrodynamic model (note $n_e = p$)

$t = 0.00$



- note that the morphology of Ω and ϕ is completely different although $\Omega = \nabla_{\perp}^2 \phi$
 $t = 23.9$



Energy Transfer in the Dissipative Coupling Model

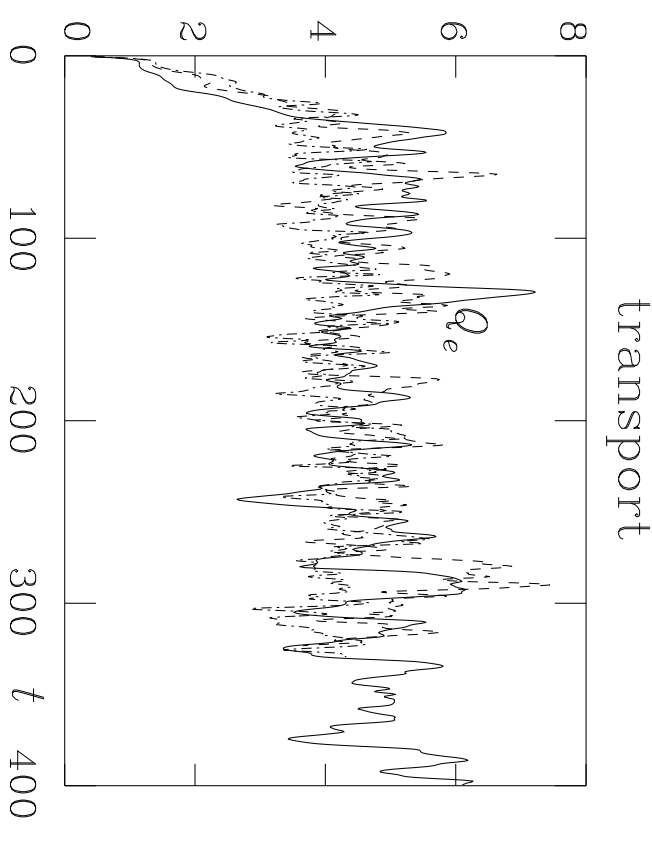
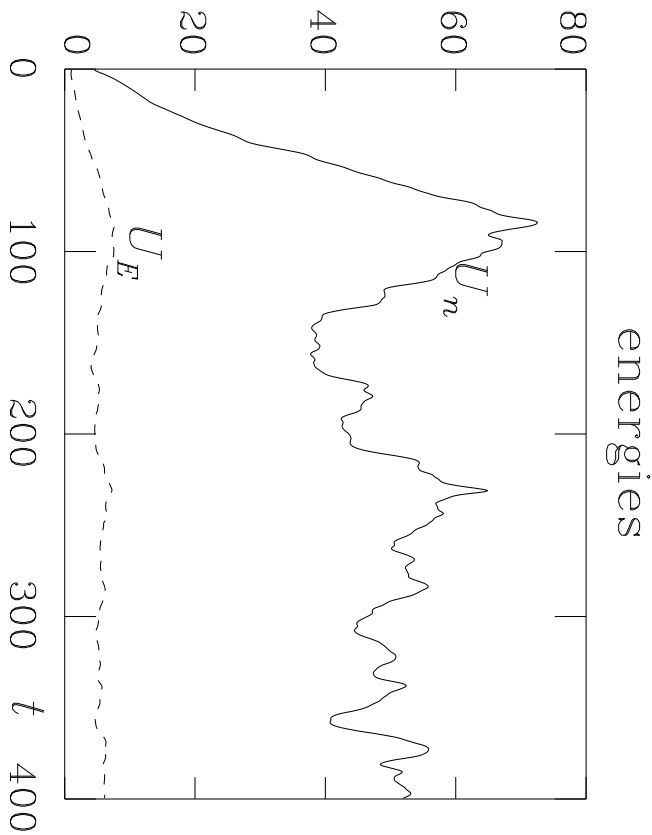
- switch gradient drive back on
- run to “saturation” defined by statistical stationarity for spectral quantities
- wide range of coupling strength, $D = 0.01, 0.03, 0.1, 0.3,$ and 1.0
- displayed for $D = 0.1$

† energy transfer directions hold for all D checked, only the robustness changes

- $D \rightarrow \infty$ is the “adiabatic limit” where $J_{\parallel} \rightarrow 0$ and $\phi \rightarrow p$

† robustness of $\mathbf{v}_E \cdot \nabla p$ proportional to about $D^{-3/4}$

- Time evolution of the dissipative coupling model, for the nominal case of $D = 0.1$



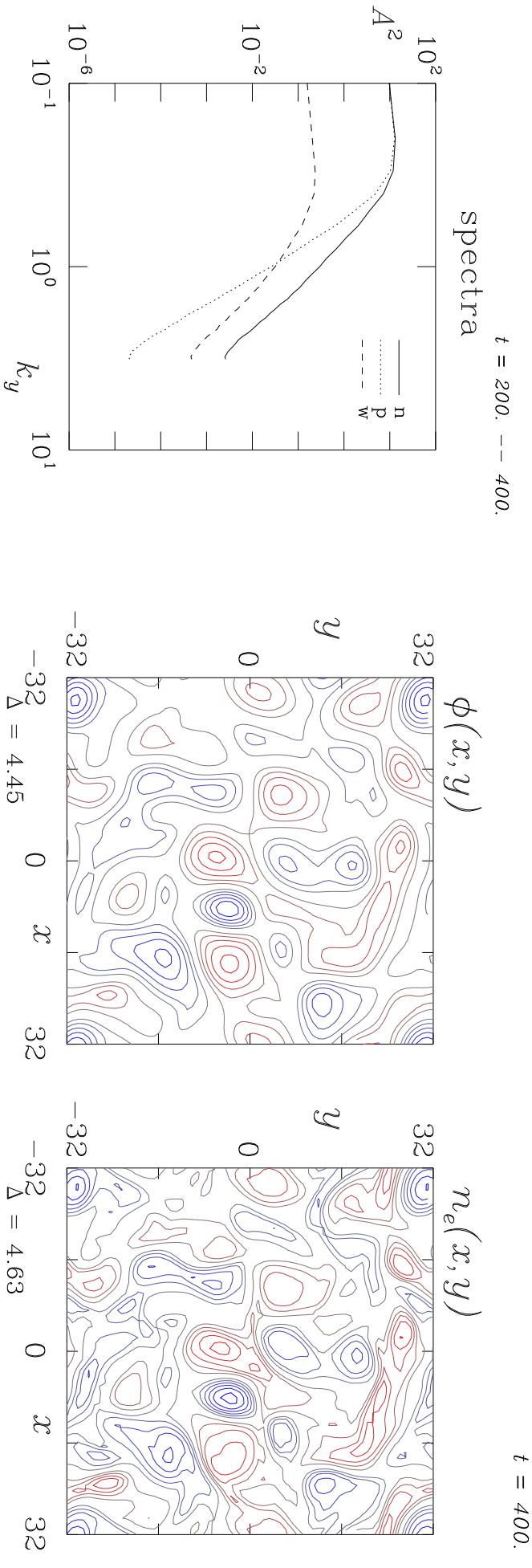
- half squared amplitudes of p and ϕ , denoted A_n and A_p , respectively

† also ExB energy (U_E) and fluctuation free energy (U_n)

- transport caused by the turbulence, $Q_e = \langle p v_E^x \rangle$, for three values of the resolution

† for 64^2 , 128^2 , and 256^2 grid nodes, the values are 4.69 ± 0.80 and 4.89 ± 0.74 and 4.14 ± 0.51

- Saturated state of the dissipative coupling model, for the nominal case of $D = 0.1$

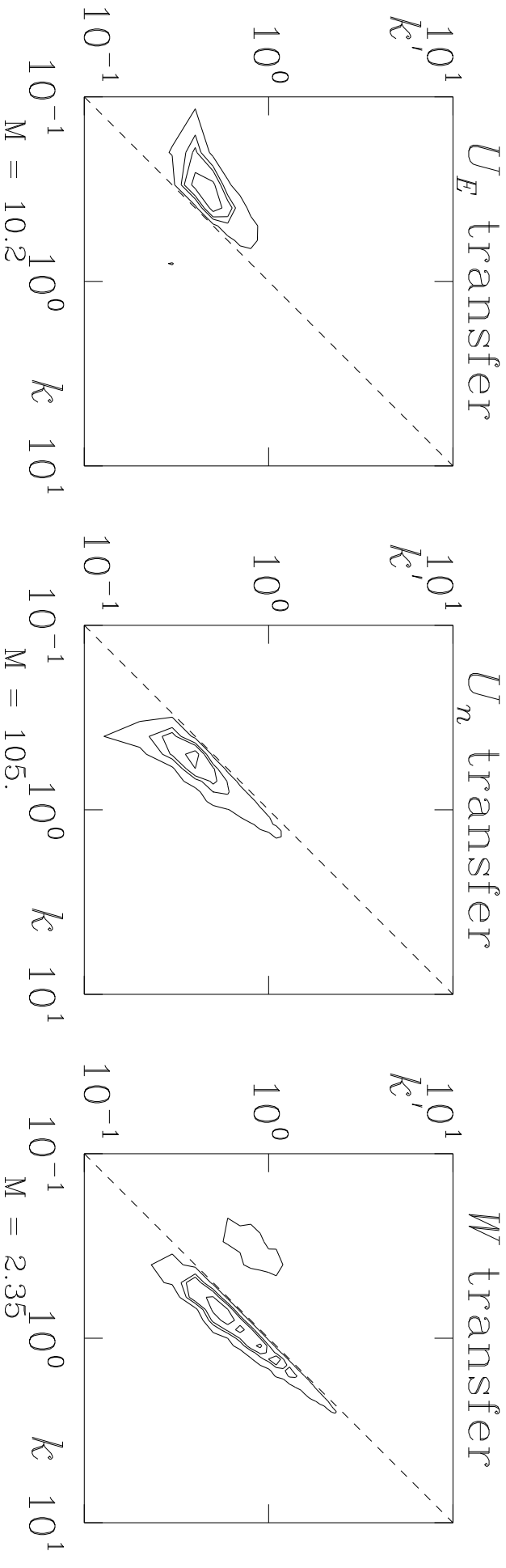


- averaged amplitude spectra for p , ϕ , and the Ω ('n', 'p', and 'w')
- morphology of ϕ and $n_e = p$ at $t = 400$

† close coupling at larger scales but differences on smaller scales, corresponding to the spectra

† the nonlinear interactions affecting p are stronger relative to the coupling at higher k_{\perp}

- Energy and enstrophy transfer in the dissipative coupling model, with $D = 0.1$



- transfer is from k' to k , shown where positive
- results show local cascade: mostly $1/2 < k'/k < 2$
- direct cascade for U_n and W , ... inverse cascade for U_E

cascade dynamics not changed by linear forcing

Transport due to ExB Turbulence

- the turbulence causes a finite average advective transport, in general ...

$$Q = Q_e + Q_i \quad Q_e = \left\langle \frac{3}{2} \tilde{p}_e v_E^x \right\rangle \quad Q_i = \left\langle \frac{3}{2} \tilde{p}_i v_E^x \right\rangle$$

- in a confined plasma, the equilibrium is maintained by a source

$$\oint dV S = \oint dS Q = \oint dV \frac{\partial Q}{\partial x}$$

- the *time scales* are very different; typical values: $\delta \sim 10^{-2}$

$$\tau_{\text{turb}} \sim 200 \frac{L_{\perp}}{c_s} \quad \tau_{\text{source}} \sim \text{few} \times \delta^{-2} \frac{L_{\perp}}{c_s}$$

- profiles evolve slowly, turbulence in quasistatic *statistical* equilibrium

it is a good approximation to consider
turbulence in the presence of a prescribed gradient