An Introduction to MHD
(or Magnetic Fluid Dynamics)

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MHD — Magnetic Fluid Dynamics

Basic Ideas, Physical Situation

- neutral, current carrying plasma
  - † ions carry mass, momentum, energy
  - † electrons carry current, thermal energy
- "slow" dynamics, finite light speed ignored

emphasise collective effects

MHD — Magnetic Fluid Dynamics

Developing MHD

- fluid ideas, constructing conservation laws
  - † material, momentum, thermal energy
  - † density, velocity, temperature (pressure)
- MHD approximation: one fluid carrying a current

Character of MHD

- magnetic flux conservation: flux tubes
- magnetic field line tension: Alfvén waves
- anisotropy: parallel and perpendicular scales
- dissipation: resistivity as magnetic diffusion

Basic Ideas of Fluid Dynamics

Continuity, Conservation Laws

- flow of material out of a fixed volume given a velocity field
  - † fixed and co-moving reference frames
  - † local transformation, advective derivative
- momentum balance
  - † pressure, self advection of velocity, Lorentz force
- heat balance
  - † 1st law of thermodynamics applied to continuum
  - † work done on fluid through compression

arrive at fluid equations, plus Maxwell equations

Meaning of Continuity

- material conservation (density, n) in presence of known flow field (velocity, $\mathbf{v}$)
- consider change of total material N in an enclosed volume V

  † for each area element ΔS, with normal $\hat{\mathbf{e}}_S$, thickness of layer exiting in time Δt is Δx

  † and number of particles in it is δN

$$\delta N = -n\,\Delta x\,\Delta S \qquad\qquad \Delta x = \mathbf{v}\cdot\hat{\mathbf{e}}_S\,\Delta t$$

- sum up area elements

$$\Delta N = \oint_V dV\,\Delta n = -\oint_S \mathbf{dS}\cdot(n\mathbf{v})\,\Delta t$$

- apply Gaussian divergence theorem, go to infinitesimal volumes and time intervals

$$\oint_V dV\,\Delta n = -\oint_V dV\,\nabla\cdot(n\mathbf{v})\,\Delta t \qquad \Longrightarrow \qquad \frac{\partial n}{\partial t} = -\nabla\cdot n\mathbf{v}$$

Divergence as Expansion of Volumes

- now take the same flow field but let area elements move with the fluid

$$\Delta N = 0 \qquad \Longrightarrow \qquad V\Delta n + n\Delta V = 0$$

- volume itself changes, and therefore so does the density

$$\frac{\Delta n}{n} = -\frac{\Delta V}{V}$$

- boundary elements move with the velocity, separation of points changes with its spatial variation

$$\Delta \mathbf{x} = \mathbf{v}\Delta t \qquad \Delta \mathbf{x}_2 - \Delta \mathbf{x}_1 = (\mathbf{v}_2 - \mathbf{v}_1)\Delta t$$

† change in volume follows (theorem from differential geometry), and hence time derivative

$$\Delta V = V\nabla\cdot(\Delta \mathbf{x}) \qquad \frac{\Delta V}{\Delta t} = V\nabla\cdot\frac{\Delta \mathbf{x}}{\Delta t} = V\nabla\cdot\mathbf{v}$$

Change in Material Density with Volume Deformation

- constant N, changing V

$$\Delta N = 0 \qquad \frac{\Delta n}{n} = -\frac{\Delta V}{V} \qquad \frac{\Delta V}{\Delta t} = V \nabla \cdot \mathbf{v}$$

- change in density with time following flow

$$\frac{dn}{dt} = -n \nabla \cdot \mathbf{v}$$

- but we also have the form using fixed volumes as a reference

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{v}$$

- we reconcile these by considering them as the same statement in different reference frames
  - † one global frame (lab frame)
  - † one local frame (near each infinitesimal volume)

The Advective Derivative

- we have the global fixed frame (t, x, y, z), and the local co-moving frame (t', x', y', z')
- consider the local transformation between them

$$dx' = dx - v_x dt \qquad dy' = dy - v_y dt \qquad dz' = dz - v_z dt$$

- the time derivative transforms as

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} \\ &= \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} \end{aligned} \qquad \Longrightarrow \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

- hence the two forms of the continuity equation are equivalent

$$\frac{dn}{dt} + n\nabla\cdot\mathbf{v} = \frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{v} = 0$$

- it is often easier to set up equation in co-moving frame and then transform back to fixed frame

Forces on the Fluid — Momentum Balance

- co-moving volume accelerated by forces

$$Nm\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

- pressure field exerts force on boundary elements, inward; apply Stokes theorem

$$\delta\mathbf{F} = -p\,\mathbf{dS} \qquad \mathbf{F} = -\oint_S \mathbf{dS}\,p \qquad \mathbf{F} = -\oint_V dV\,\nabla p$$

- put in electric force, go to infinitesimal volumes

$$nm\frac{d\mathbf{v}}{dt} = -\nabla p + nq\mathbf{E}$$

- now transform to fixed frame, *note it is a Lorentz transformation* for $\mathbf{E}$

$$nm\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + nq\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)$$

- main differences to particle motion are pressure, velocity self advection

Thermodynamics — Energy Conservation

- velocity divergence implies work done on or by co-moving volumes
- apply simple thermodynamics, neglect heat transfer through boundary

$$\Delta E + p\,\Delta V = \Delta Q \rightarrow 0 \qquad\qquad \frac{dE}{dt} + p\frac{dV}{dt} = 0$$

- ideal gas law

$$\Delta E = \frac{3}{2} N k_B \Delta T$$

- go to infinitesimal volumes

$$\frac{3}{2} n k_B \frac{dT}{dt} + p \nabla \cdot \mathbf{v} = 0$$

- as pressure equation, $p = n k_B T$, with general ratio sp. heats Γ; transform to fixed frame

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \Gamma\, p \nabla \cdot \mathbf{v} = 0$$

Plasma Electrodynamics

- one set of fluid equations for each charged particle species

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{v} = 0$$

$$nm\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + nq\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \Gamma\, p\nabla\cdot\mathbf{v} = 0$$

- plus Maxwell's equations

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0 \qquad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} = \nabla\times\mathbf{B}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\cdot\mathbf{E} = 4\pi\rho_{ch}$$

- charge density and current are summed over species

$$\rho_{ch} = \sum_\alpha n_\alpha q_\alpha \qquad \mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{v}_\alpha$$

E Pluribus Unum

the steps to MHD

- low frequencies (compared to light waves)
  - † static Ampere's law, slow speed limit of Lorentz gauge
  - † neutral plasma, "quasineutral" dynamics
- down to one fluid
  - † pressure force smaller than electric force
  - † current as relative drift smaller than fluid velocity

single total pressure, single velocity

Low Frequency Dynamics

- combine Maxwell $\mathbf{E}$ and $\mathbf{B}$ equations, either satisfy in vacuum (zero divergences)

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\nabla\times(\nabla\times\mathbf{B}) = \nabla^2\mathbf{B}$$

- frequency and wavenumber scaling (inverse time/space scales)

$$\frac{\partial^2}{\partial t^2} \to -\omega^2 \qquad\qquad \nabla^2 \to -k^2$$

- "low frequencies" means $\omega \ll kc$

  † hence neglect $c^{-1}(\partial/\partial t)$ terms (except $\partial\mathbf{B}/\partial t$ since $\nabla\times\mathbf{E}$ is in general nonzero)

  † basic result: divergence free current

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

becomes $\quad \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} \qquad$ hence $\quad \nabla\cdot\mathbf{J} = 0$

- with $\nabla\cdot\mathbf{J}=0$ the following all go together

$$\nabla\cdot\mathbf{J}\to 0 \quad\Longrightarrow\quad \left\{\begin{array}{r}\dfrac{1}{c}\dfrac{\partial\mathbf{E}}{\partial t}\ll\dfrac{4\pi}{c}\mathbf{J}\\ \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{A}}{\partial t^2}\ll\nabla^2\mathbf{A}\\ \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}\ll\nabla^2\phi\\ \dfrac{1}{c}\dfrac{\partial\phi}{\partial t}\ll\nabla\cdot\mathbf{A}\\ \rho_{ch}\ll n_e e\end{array}\right.$$

- note that $\nabla\cdot\mathbf{E}$ can still be nonzero, with vortical motion

$$\nabla\times\mathbf{v}=\nabla\times\left(c\,\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\qquad\text{e-static with}\quad\mathbf{E}=-\nabla\phi\qquad\Longrightarrow\qquad\frac{c\mathbf{B}}{B^2}\nabla^2\phi$$

† we simply have the attendant charge density implied by $\nabla\cdot\mathbf{E}$ negligibly small

$$\frac{T_e}{4\pi n_e e^2}\nabla^2\frac{e\phi}{T_e}=-\frac{\rho_{ch}}{n_e e}\ll 1\qquad\Longrightarrow\qquad k_\perp^2\lambda_D^2(e\phi/T_e)\ll 1\qquad\text{where}\quad\lambda_D^2=\frac{T_e}{4\pi n_e e^2}$$

A Single Fluid

- we have for each fluid, Lorentz force in $\mathbf{v}$ equation and velocity divergences in p equation

  † basic requirement is that no species (sp. electrons) is “special” with regard to p or $\mathbf{v}$

$$\rho_{ch}\mathbf{E} \ll \frac{\mathbf{J}\times\mathbf{B}}{c} \qquad n_e m_e \ll n_i M_i$$

$$\frac{\mathbf{J}}{n_e e} \ll \mathbf{v}_i \approx \mathbf{v}_e \qquad \frac{\nabla p_e}{n_e e} \ll \mathbf{E} \approx -\frac{\mathbf{v}}{c}\times\mathbf{B}$$

  † required assumptions: quasineutrality (easy), small mass ratio (easy), ...

  † last two must be checked *a posteriori* (not so easy)

- single density is summed mass density; single charge density is zero
- single velocity is the ExB velocity
- single pressure is summed total pressure

$$\rho = \sum_\alpha n_\alpha m_\alpha \qquad \mathbf{v} = \mathbf{v}_E = c\,\frac{\mathbf{E}\times\mathbf{B}}{B^2} \qquad p = \sum_\alpha p_\alpha$$

MHD Kinematic and Force Equations

- all Lorentz forces go to zero $\rightarrow$ ExB velocity

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E} \qquad \Longrightarrow \qquad \frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})$$

MHD kinematic equation

- add up all Lorentz forces (lowest order ExB effects cancel)

$$\mathbf{F}_L = \left[\sum_\alpha n_\alpha q_\alpha\right]\mathbf{E} + \frac{1}{c}\left[\sum_\alpha n_\alpha q_\alpha \mathbf{v}_\alpha\right]\times\mathbf{B} \qquad \Longrightarrow \qquad \rho\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{\mathbf{J}\times\mathbf{B}}{c}$$

MHD force equation

MHD at a Glance

- equations for density, velocity, pressure, and magnetic field

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \Gamma \, p \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- current given by Ampere's law

$$\frac{4\pi}{c} \mathbf{J} = \nabla \times \mathbf{B}$$

- note the electric field has submerged, being given by the ExB velocity

$$\mathbf{v} = c \, \frac{\mathbf{E} \times \mathbf{B}}{B^2} \qquad\qquad \mathbf{E} = -\frac{\mathbf{v} \times \mathbf{B}}{c}$$

Part II — The Character of MHD

- magnetic flux conservation: flux tubes
- magnetic field line tension: Alfvén waves
- anisotropy: parallel and perpendicular scales
- dissipation: resistivity as magnetic diffusion

Magnetic Flux Conservation

Frozen-in Field Lines

- magnetic flux defined by magnetic field projected onto an area

$$\Phi = \oint_S \mathbf{B} \cdot \mathbf{dS}$$

- consider $\mathbf{S}$ as an infinitesimal, arbitrary surface co-moving with the flow

$$\Phi = \mathbf{B} \cdot \mathbf{S}$$

- find the co-moving time derivative of the flux

$$\frac{d\Phi}{dt} = \frac{d\mathbf{B}}{dt} \cdot \mathbf{S} + \mathbf{B} \cdot \frac{d\mathbf{S}}{dt}$$

- much of this is geometry:

  † how does a directed area change with an arbitrary displacement field

deformation of the area

- the co-moving area changes as

$$\frac{d\mathbf{S}}{dt} = -(\mathbf{S}\times\nabla)\times\mathbf{v} = \mathbf{S}\nabla\cdot\mathbf{v} - (\nabla\mathbf{v})\cdot\mathbf{S}$$

- proof: decompose **S** into triangles, find motion of vertices, using cross product of legs for **dS**

$$\frac{d}{dt}\mathbf{dS} = (\mathbf{x}_1 - \mathbf{x}_0)\times\frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) + \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_0)\times(\mathbf{x}_2 - \mathbf{x}_1)$$

$$\frac{d}{dt}\mathbf{dS} = -(\mathbf{x}_1 - \mathbf{x}_0)\times\frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_0) - (\mathbf{x}_2 - \mathbf{x}_1)\times\frac{1}{2}(\mathbf{v}_2 + \mathbf{v}_1) - (\mathbf{x}_0 - \mathbf{x}_2)\times\frac{1}{2}(\mathbf{v}_0 + \mathbf{v}_2)$$

$$\frac{d}{dt}\mathbf{dS} = -\sum_j \Delta\mathbf{x}_j\times\mathbf{v}_j = -\oint_L \mathbf{dl}\times\mathbf{v}$$

and after another vector/manifold theorem

$$\frac{d}{dt}\mathbf{dS} = -(\mathbf{dS}\times\nabla)\times\mathbf{v}$$

$$\mathbf{S} = \sum_{\text{pieces}} \mathbf{dS} \qquad \Longrightarrow \qquad \frac{d\mathbf{S}}{dt} = -(\mathbf{S}\times\nabla)\times\mathbf{v}$$

magnetic flux conservation, completion

- find the co-moving time derivative of the flux

$$\frac{d\Phi}{dt} = \frac{d\mathbf{B}}{dt} \cdot \mathbf{S} + \mathbf{B} \cdot \frac{d\mathbf{S}}{dt}$$

- advective derivative for magnetic field, co-moving deformation of area

$$\frac{d\mathbf{B}}{dt} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} \qquad \frac{d\mathbf{S}}{dt} = \mathbf{S} \nabla \cdot \mathbf{v} - (\nabla \mathbf{v}) \cdot \mathbf{S}$$

- substitute these forms in … end with MHD kinematic equation

$$\frac{d\Phi}{dt} = \left[\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B}\right] \cdot \mathbf{S} + \mathbf{B} \cdot \left[\mathbf{S} \nabla \cdot \mathbf{v} - (\nabla \mathbf{v}) \cdot \mathbf{S}\right]$$

$$\frac{d\Phi}{dt} = \left[\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B}(\nabla \cdot \mathbf{v}) - \mathbf{B} \cdot \nabla \mathbf{v}\right] \cdot \mathbf{S}$$

$$\frac{d\Phi}{dt} = \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B})\right] = 0 \qquad \text{QED}$$

The Idea of a Flux Tube

- construction of a flux tube
  - † draw a closed curve in the plane locally perpendicular to $\mathbf{B}$
  - † transport each point on the curve a given distance along $\mathbf{B}$
  - † join the transported points into another closed curve
  - † the field lines between the points form the surface of the flux tube
  - † the areas in the two perpendicular planes enclosed by the curves form the ends of the flux tube
- by flux conservation . . .

flux tubes cannot pass through each other

- consequence: individual field lines have their identities topologically preserved
  - † "frozen-in field lines" picture (broken by finite plasma resistivity)

Magnetic Pressure and Tension

- form the J-cross-B, or “MHD” force into two pieces

$$\frac{1}{c}\,\mathbf{J}\times\mathbf{B} = \frac{1}{c}\,\left(\frac{c}{4\pi}\,\nabla\times\mathbf{B}\right)\times\mathbf{B} = -\nabla\frac{B^2}{8\pi} + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}$$

- “magnetic pressure” — scalar piece combines with p

$$-\nabla\left(p + \frac{B^2}{8\pi}\right)$$

- “magnetic tension” — vector piece ↔ restoration against curvature — “field line bending”

$$\mathbf{B}\cdot\nabla\mathbf{B} \approx -\frac{B^2}{r}\,\hat{\mathbf{e}}_r$$

† force is directed towards center of curvature $\Longrightarrow$ field lines as “wires with tension”

mag. tension: origin of basic character of MHD

Shear Alfvén Waves

- incompressible motion: not only $\nabla \cdot \mathbf{B}$ but also $\nabla \cdot \mathbf{v}$ vanishes
  - † shearing motion across magnetic field lines, $\mathbf{v}_\perp$
  - † disturbance perpendicular to magnetic field, $\mathbf{B}_\perp$
  - † wavelike propagation along the magnetic field, $\partial/\partial t$ and $\mathbf{B} \cdot \nabla$
  - † not unlike waves on a string with tension
- disturbances propagate at the “Alfvén velocity” defined by

$$\boxed{v_A = B/\sqrt{4\pi\rho}}$$

- most basic dynamical difference between MHD and hydrodynamics

shear Alfvén waves, detail

- small disturbances on a homogeneous, magnetised plasma at rest

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_\perp \qquad \mathbf{B}\cdot\nabla\mathbf{B} \to (\mathbf{B}_0\cdot\nabla)\mathbf{B}_\perp \qquad \mathbf{B}\cdot\nabla\mathbf{v} \to (\mathbf{B}_0\cdot\nabla)\mathbf{v}_\perp$$

- perpendicular motion, no divergence $\Longrightarrow$ MHD kinematic equation becomes

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) \qquad \Longrightarrow \qquad \frac{\partial\mathbf{B}_\perp}{\partial t} = (\mathbf{B}_0\cdot\nabla)\mathbf{v}_\perp$$

- no divergence $\Longrightarrow$ no change in ρ, p, or B^2 at linear order, so MHD force equation becomes

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi} \qquad \Longrightarrow \qquad \frac{\partial\mathbf{v}_\perp}{\partial t} = \frac{1}{4\pi\rho}(\mathbf{B}_0\cdot\nabla)\mathbf{B}_\perp$$

- the two equations form a wave system, defining the “Alfvén velocity”

$$\frac{\partial^2}{\partial t^2}\begin{bmatrix}\mathbf{B}_\perp\\ \mathbf{v}_\perp\end{bmatrix} = \frac{1}{4\pi\rho}(\mathbf{B}_0\cdot\nabla)^2\begin{bmatrix}\mathbf{B}_\perp\\ \mathbf{v}_\perp\end{bmatrix} \qquad \text{with} \qquad V_{ph}^2 = \frac{B_0^2}{4\pi\rho} \qquad \text{defining} \qquad v_A = \frac{B_0}{\sqrt{4\pi\rho}}$$

Magnetic Pressure Waves

- small compressional disturbances, one dimensional

  † motion is across field lines, $\mathbf{v}_\perp$

  † magnetic disturbance is parallel, with $\widetilde{B}^2 = 2\mathbf{B}_0 \cdot \widetilde{\mathbf{B}}$

  † hence $\mathbf{B} \cdot \nabla$ effects are not involved

- MHD force and kinematic equations, linearised (hence constant density)

$$\rho \frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}_\perp) = -\nabla^2 \left(p + \frac{B^2}{8\pi} \right) \qquad \frac{\partial}{\partial t} \left(p + \frac{B^2}{8\pi} \right) = -\left(\Gamma\, p + \frac{B^2}{4\pi} \right) \nabla \cdot \mathbf{v}_\perp$$

- compressional wave speed, V_a, is given by the sound and Alfvén velocities

$$V_a^2 = \Gamma\, \frac{p}{\rho} + \frac{B^2}{4\pi\rho} = v_s^2 + v_A^2$$

- compressibility controlled by plasma or magnetic field, according to the larger of the two

  † the parameter controlling this is . . .

The Plasma Beta

- compressional forces, gradients of scalars

$$\nabla^2 \left(p + \frac{B^2}{8\pi} \right) \qquad \text{involved in} \qquad \frac{\partial}{\partial t}(\nabla \cdot \mathbf{v})$$

- ratio of plasma pressure to magnetic pressure defines the "plasma beta"

$$\boxed{\beta = 8\pi\, p/B^2}$$

- for "low beta" dynamics, $\beta \ll 1$, pressure and density can evolve independently

  † while magnetic pressure maintains incompressibility of perpendicular dynamics

  † picture holds as long as

$$v_A \nabla_\perp \gg \frac{\partial}{\partial t}$$

  † note that *quasistatic* compressibility can remain in an inhomogeneous magnetic field

Basic Time Scales of MHD

- usual equilibrium is anisotropic, with parallel and perpendicular length scales, $L_\parallel$ and $L_\perp$

$$L_\perp \ll L_\parallel$$

- system size time scales

  † shear Alfvén time scale is $L_\parallel / v_A$

  † for low beta, compressional time scale is $L_\perp / v_A$

- corresponding dynamical frequencies follow typical wavenumbers of global or micro disturbances

$$k_\parallel \sim \frac{2\pi}{L_\parallel} \qquad\qquad k_\perp \sim \frac{2\pi}{\Delta_\perp}$$

- usual ordering is

$$\omega \sim k_\parallel v_A \ll k_\perp v_A$$

low beta dynamical incompressibility

Dissipation — Effect of Finite Resistivity

- correction to electron force balance (assume isotropic, constant resistivity given by η)

$$\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B} = \eta\mathbf{J}$$

- correction to MHD kinematic equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B} - c\eta\mathbf{J})$$

- insert Ampere's law $\mathbf{J} = (c/4\pi)\nabla\times\mathbf{B}$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \frac{\eta c^2}{4\pi}\nabla^2\mathbf{B}$$

resistivity gives diffusivity of magnetic field lines

- on a small scale with $(\eta c^2/4\pi)k_\perp^2 \sim \omega$ field lines slip past the fluid, lose topological identity

Checking the Validity of MHD

- most subtle limitation is the one fluid approximation(s)

$$\nabla p_e \ll n_e e \mathbf{E} \qquad\qquad \mathbf{J} \ll n_e e\, \mathbf{v}_e$$

- both are related, follow comparison of static parallel forces

$$E_\parallel = -\frac{1}{c}\frac{\partial A_\parallel}{\partial t} - \nabla_\parallel \phi \qquad\qquad \text{consider} \qquad \nabla_\parallel p_e + n_e e \nabla_\parallel \phi$$

† electron force potential is given by pressure and electrostatic potential

MHD is valid only if $\nabla_\parallel p_e \ll n_e e \nabla_\parallel \phi$

- for dynamics in magnetically confined plasmas MHD requires

$$\widetilde{p}_e \ll n_e e\, \widetilde{\phi}$$

Limitation of MHD for Magnetically Confined Plasmas

- dynamics driven by magnetic field is almost always "safe"

  † "low" beta, "large" scale hence "small" $k_\perp$, "fast" dynamics (follows v_A)

- main subtlety is gradient driven dynamics

  † $k_\| L_\| \sim 2\pi$ set by geometry

  † frequencies set by advection of pressure disturbances along pressure gradient

- the general frequency limitation of MHD following this is (details)

$$\omega \gg \omega_* = \frac{ck_\perp T_e}{eBL_\perp}$$

- most unfortunate complication is that one must go outside MHD to get this result . . .

$\Longrightarrow$ "two fluid dynamics" (cf. References)